

Generic transversality property for a class of wave equations with variable damping

Romain JOLY

Université Paris Sud,

Analyse Numérique et EDP, UMR 8628

Bâtiment 425,

F-91405 Orsay Cedex, France

Romain.Joly@normalesup.org

Abstract : *In this paper, we focus on the one-dimensional wave equations with variable damping, either in the interior or on the boundary of an interval. We show that, generically with respect to the non-linearity, they satisfy the Kupka-Smale property, that is that all their equilibrium points are hyperbolic and that their stable and unstable manifolds intersect transversally.*

More generally, we show that the generic Kupka-Smale property holds for an abstract equation

$$u_{tt} + B(u + \Gamma u_t) = f(x, u) , \quad (0.1)$$

for which we assume, besides other hypotheses, that the rootvectors of the associated linear operator form a Riesz basis. We notice that other one-dimensional damped equations fit into the frame of Equation (0.1). Additional qualitative properties of (0.1) are also obtained.

Keywords : dissipative wave equations, hyperbolicity, transversality, Morse-Smale property, Kupka-Smale property, generic results.

Résumé : *Dans cet article, nous nous intéressons aux équations des ondes avec dissipation variable à l'intérieur ou au bord d'un intervalle. Nous montrons que, génériquement par rapport à la non-linéarité, elles possèdent la propriété de Kupka-Smale, c'est-à-dire que leurs points d'équilibre sont hyperboliques et que leurs variétés stables et instables sont*

transverses.

Plus généralement, nous montrons que cela est vrai pour une équation abstraite du type

$$u_{tt} + B(u + \Gamma u_t) = f(x, u) , \quad (0.2)$$

pour laquelle on suppose, entre autres, que les vecteurs propres généralisés de l'opérateur linéarisé associé forment une base de Riesz. L'équation (0.2) modélise d'autres équations des ondes dissipatives en dimension un. Nous montrons au passage différentes propriétés qualitatives de l'équation (0.2).

Mots Clés : équations des ondes dissipatives, hyperbolicité, transversalité, propriété de Morse-Smale, propriété de Kupka-Smale, résultats génériques.

1 Introduction

In the qualitative study of partial differential equations, the Morse-Smale property plays an important role as it ensures in some sense the stability of the qualitative behaviour of solutions. It is possible to define the Morse-Smale property for general dynamical systems. However, since few facts are known concerning non-gradient Morse-Smale systems, even in the finite-dimensional case, we will restrict ourselves to gradient systems, that is to dynamical systems, for which there exists a strict Lyapunov function. We recall that the ω -limit set of any compact trajectory of a gradient system consists only on equilibrium points. A gradient system is said to have the Morse-Smale property if it has a finite number of equilibria, which are all hyperbolic, and if all the stable and unstable manifolds of the equilibria intersect transversally (we shall recall these notions before introducing Theorem 1.1 ; for more details, see [18] or [32]). For gradient systems on finite-dimensional compact manifolds, it is well-known that the Morse-Smale property implies the stability of the system, that is that, for small perturbations of the vector-field, the flow remains qualitatively the same. The Kupka-Smale theorem implies that the Morse-Smale property is generic in the class of gradient systems on finite-dimensional compact manifolds (see [32], p.152). We recall that a set is said to be generic if it contains a countable intersection of open dense sets, and that a property is said to be generic if it holds on a generic set. The genericity of a property in Banach spaces implies that this property holds on a dense set ; it corresponds to the “almost everywhere” notion in measurable spaces. So we can say that almost all the gradient systems on finite-dimensional compact manifolds have the Morse-Smale property.

It is therefore natural to wonder what can be generalised to infinite-dimensional gradient systems, in particular to those generated by partial differential equations. This is not only a theoretical question. Indeed, when one works for example on a numerical simulation, or on a physical system with parameters, which were imprecisely measured, one deals with a perturbation of the original system. If the last one is stable, one can consider that the qualitative behaviour, which is observed on the approximative system, holds for the exact one.

In general, infinite-dimensional gradient systems, and also gradient systems defined on finite-dimensional non-compact manifolds, have an infinite number of equilibria. For this reason, we consider here the Kupka-Smale property. A gradient system is said to have the Kupka-Smale property if all its equilibria are hyperbolic and all its stable and unstable manifolds intersect transversally. Under additional dissipative hypotheses, all the equilibria of a Kupka-Smale gradient system S belong to a compact set, which implies that the system is Morse-Smale. In particular, if a Kupka-Smale gradient system S has a compact global attractor \mathcal{A} , then it is Morse-Smale. The structural stability result concerning Morse-Smale systems on finite-dimensional compact manifolds has been extended as follows by Oliva (see [18]). Let $S_\epsilon(t)$, $t \in \mathbb{R}_+$, be a parametrized family of gradient systems, having a compact global attractor \mathcal{A}_ϵ on X . If the attractors \mathcal{A}_ϵ are upper-semicontinuous in ϵ at $\epsilon = 0$, and if $S_0(t)$, $t \in \mathbb{R}_+$, is Morse-Smale, then, under additional reversibility hypotheses, the restrictions to \mathcal{A}_ϵ of the discrete dynamical systems $(S_\epsilon(1))^n$ are conjugated to the restriction to \mathcal{A}_0 of $(S_0(1))^n$, for $\epsilon > 0$ small enough. More precisely, there exists a diffeomorphism h_ϵ from \mathcal{A}_0 onto \mathcal{A}_ϵ such that $h_\epsilon \circ (S_0(1))^n = (S_\epsilon(1))^n \circ h_\epsilon$. This property implies in particular that $S_\epsilon(t)$, $t \in \mathbb{R}_+$, is still a Morse-Smale system and that its phase-diagram (that is the description of its equilibria and the trajectories connecting them) is equivalent to the one of $S_0(t)$. This shows that the Morse-Smale property is still relevant for infinite-dimensional gradient systems.

Next, one can wonder if the property of genericity of the gradient Kupka-Smale systems on finite-dimensional manifolds extends to the infinite-dimensional case in a meaningful way. The first results on genericity of the Kupka-Smale property for partial differential equations concern the parabolic equation

$$\begin{cases} u_t(x, t) = \Delta u(x, t) + f(x, u(x, t)) , & (x, t) \in \Omega \times \mathbb{R}_+ \\ u(x, t) = 0 , & (x, t) \in \partial\Omega \times \mathbb{R}_+ \end{cases} , \quad (1.1)$$

where $\Omega \subset \mathbb{R}^N$. Henry proved that in space dimension $N = 1$, a heteroclinic orbit connecting two equilibria is necessarily transversal (see [23] and also [1]). This implies the genericity of the Kupka-Smale property, since the equilibria of (1.1) are all hyperbolic generically with respect to the non-linearity f (see [4]). Unfortunately, this transversality property is not true in higher dimension. Brunovský and Poláčik proved that the equation (1.1) satisfies the Kupka-Smale property generically with respect to the non-linearity f ,

for any dimension $N \geq 1$ (see [5]).

Later, Brunovský and Raugel considered the wave equation with constant damping

$$\begin{cases} u_{tt}(x, t) + \gamma u_t(x, t) = \Delta u(x, t) + f(x, u(x, t)) , & (x, t) \in \Omega \times \mathbb{R}_+ \\ u(x, t) = 0 , & (x, t) \in \partial\Omega \times \mathbb{R}_+ \end{cases} , \quad (1.2)$$

where $\Omega \subset \mathbb{R}^N$ and γ is a positive constant.

They proved that this equation satisfies the Kupka-Smale property generically with respect to the non-linearity f , for any dimension $N \geq 1$ (see [6]). Notice that, as the dynamics of hyperbolic equations are richer than the ones of parabolic equations, the transversality result of Henry is no longer true even in the one-dimensional case. So, the genericity result of [6] is also meaningful in this case. In their proofs, Brunovský and Poláčik used an abstract genericity theorem, that Brunovský and Raugel improved, in order to be able to apply it in the hyperbolic case. We recall this improved version in Appendix A. This theorem is very useful for showing genericity of the Kupka-Smale property for partial differential equations with respect to a class of non-linearities depending on a parameter. The key points of its proof are a version of the Sard-Smale theorem (a slightly stronger version than Theorem A.2 given here), and a functional characterisation of the transversality. Applying this abstract theorem, Brunovský and Poláčik, as well as Brunovský and Raugel, reduced their genericity problem to the construction of a perturbation $h(x, u)$ of the nonlinearity $f(x, u)$, such that a certain integral, depending on h , the heteroclinic orbits and the global bounded solutions of the adjoint linearized equation to (1.1) and (1.2) respectively, does not vanish (see the integrals I and J given by (5.7) and (5.8) in our case). In both papers, the suitable perturbation $h(x, u)$ is localized in the neighborhood of an appropriate point $x = x_0$ of the domain Ω . In [5], the choice of x_0 was the consequence of delicate properties of the singular nodal set of the reaction-diffusion equation. Since corresponding nodal properties are not available in the case of the hyperbolic equation, other techniques had to be used. The proof in [6] uses the development of the respective globally defined and bounded solutions $u(x_0, t)$ and $\psi(x_0, t)$, of the equation (1.2) and its corresponding adjoint linearized equation, into fractional series in the neighborhood of certain points and also their asymptotic development in the neighborhood of $t = -\infty$. The asymptotic development of these bounded solutions strongly depends on spectral properties of the linearized equations around the equilibria.

A natural extension of the above results is the study of wave equations with non-constant damping. As said above, spectral properties of the equation play an important role in the proof of [6]. In particular, the existence of a Riesz basis composed by the eigenvectors of the linearized operator is used there. The existence of such a Riesz basis is known in general only in the one-dimensional case. For this reason, we restrict our study here to this case. The most classical example of wave equation with non-constant damping

is the wave equation with internal local damping

$$\begin{cases} u_{tt}(x, t) + \gamma(x)u_t(x, t) = u_{xx}(x, t) + f(x, u(x, t)) , & (x, t) \in [0, 1] \times \mathbb{R}_+ \\ u(x, t) = 0 , & (x, t) \in \{0, 1\} \times \mathbb{R}_+ \end{cases} \quad (1.3)$$

where γ is a nonnegative bounded function which is positive on an open subset of $[0, 1]$. This equation generates a dynamical system $S(t) : U_0 \in X \mapsto S(t)U_0$ on the Banach space $X = \mathbb{H}_0^1([0, 1]) \times \mathbb{L}^2([0, 1])$. An equilibrium E of (1.3) is said hyperbolic if the spectrum of the linearization $D_U S(1)E$ does not intersect the unit circle in the complex plane. We also recall that two submanifolds of X intersect transversally if, at any point of intersection, one of the two tangent spaces contains a closed complement of the other. For $k \geq 1$, we denote \mathfrak{G}^k the set $\mathcal{C}^k([0, 1] \times \mathbb{R}, \mathbb{R})$ endowed with the Whitney topology, that is, the topology generated by the sets

$$\{g \in \mathfrak{G}^k / |D^i f(x, u) - D^i g(x, u)| < \delta(u), i = 0, \dots, k, x \in [0, 1], u \in \mathbb{R}\},$$

where δ is any positive function on \mathbb{R} and f is any function of $\mathcal{C}^k([0, 1] \times \mathbb{R}, \mathbb{R})$. A sequence of functions f_n converges to f in the Whitney topology if and only if there exists a compact set $K \subset \mathbb{R}$ such that for all $i = 0, \dots, k$, the derivatives $D^i f_n$ converge uniformly to $D^i f$ on $[0, 1] \times K$, and for all n , except maybe a finite number, $f_n = f$ on $[0, 1] \times (\mathbb{R} \setminus K)$. Moreover, the space \mathfrak{G}^k is a Baire space, which means that a generic set of \mathfrak{G}^k is also a dense set. For more details concerning this topology, see [15].

One of the main results of this paper is the following theorem.

Theorem 1.1. *Let $k \geq 2$ and let \mathfrak{G}^{KS} be the set of functions $f \in \mathfrak{G}^k$ such that the damped wave equation (1.3) satisfies the Kupka-Smale property, that is, such that all the equilibria of (1.3) are hyperbolic and their stable and unstable manifolds intersect transversally. Then \mathfrak{G}^{KS} is a generic subset of \mathfrak{G}^k .*

Restricting the above space \mathfrak{G}^k , we show the genericity of the Morse-Smale property. For example, let $\mathfrak{G}_{\text{diss}}^k$ be the open subset of \mathfrak{G}^k defined by

$$\mathfrak{G}_{\text{diss}}^k = \{f \in \mathfrak{G}^k / \limsup_{u \rightarrow \pm\infty} \frac{f(x, u)}{u} < 0\} .$$

When f belongs to $\mathfrak{G}_{\text{diss}}^k$, we show in Corollary 2.7 that Equation (1.3) admits a compact global attractor. Thus, the number of equilibria is finite and the Kupka-Smale property is equivalent to the Morse-Smale one.

Corollary 1.2. *If $k \geq 2$, the damped wave equation (1.3) satisfies the Morse-Smale property for a generic dissipative non-linearity $f \in \mathfrak{G}_{\text{diss}}^k$.*

The knowledge of the asymptotic behaviour of the spectrum of the linear operator associated to the wave equation with constant damping (1.2) is a key-point in the proof of [6]. In the constant damping case, explicit relations between the eigenvalues and the eigenfunctions of the damped wave operator and those of the Laplacian operator are known. In particular, the eigenvalues are either real or belong to the same vertical line. In the case of the one-dimensional wave equation with non-constant damping (1.3), one only knows that the generalized eigenvectors of the operator form a Riesz basis and that the real part of the eigenvalues have only one point of accumulation. As these spectral properties are weaker, the proof of Theorem 1.1 is more involved. We emphasize that the Riesz basis property is central in our proofs. This property is strongly linked to the dimension one as it is known not to hold in general in higher dimensional cases. The few examples in higher dimensions in which the eigenvectors form a Riesz basis are very particular (e.g. the constant damping case or equation with radial symmetry, see [36]). For these reasons we only consider here the one dimensional case.

Another important example of hyperbolic equation with non-constant damping is the following wave equation damped on the boundary,

$$\begin{cases} u_{tt}(x, t) = u_{xx}(x, t) - u(x, t) + f(x, u(x, t)) , & (x, t) \in [0, 1] \times \mathbb{R}_+ \\ \frac{\partial u}{\partial \nu}(x, t) + \gamma(x)u_t(x, t) = 0 , & (x, t) \in \{0, 1\} \times \mathbb{R}_+ \end{cases} \quad (1.4)$$

where γ is a nonnegative bounded function which is positive on at least one point of $\{0, 1\}$. When $\gamma(0) \neq 1$ and $\gamma(1) \neq 1$, the structure of the spectrum of the linearized operator is similar to the one of Equation (1.3). Thus, we obtain the following result.

Theorem 1.3. *If we assume that $k \geq 2$, $\gamma(0) \neq 1$ and $\gamma(1) \neq 1$, then the set of functions $f \in \mathfrak{G}^k$, such that Equation (1.4) has the Kupka-Smale property, is a generic subset of \mathfrak{G}^k . Moreover, the set of functions $f \in \mathfrak{G}_{diss}^k$, such that Equation (1.4) has the Morse-Smale property, is a generic subset of \mathfrak{G}_{diss}^k .*

When $\gamma(0) = 1$ or $\gamma(1) = 1$, the spectrum of the linearized operator obtained by choosing $f = 0$ is empty. See Subsection 3.1 for a discussion of this case.

In order to enhance the common structures of Equations (1.3) and (1.4) and to make easier the understanding of the mechanism of the proofs, we have chosen to work with an abstract wave equation

$$\frac{\partial}{\partial t} \begin{pmatrix} u \\ u_t \end{pmatrix} = A \begin{pmatrix} u \\ u_t \end{pmatrix} + \begin{pmatrix} 0 \\ f(x, u) \end{pmatrix} = \begin{pmatrix} u_t \\ -B(u + \Gamma u_t) \end{pmatrix} + \begin{pmatrix} 0 \\ f(x, u) \end{pmatrix}, \quad (1.5)$$

where the operators B and Γ will be defined in Section 2. We assume that the eigenvectors of A form a Riesz basis. Adding other hypotheses, we prove the genericity of Kupka-Smale property for Equation (1.5) (see Theorem 2.6). Theorems 1.1 and 1.3 will be a direct

consequences of this abstract theorem. Moreover, Theorem 2.6 can be applied to other one-dimensional dissipative wave equations.

Notice that, in showing the genericity of the transversality, we need to prove auxiliary properties, which have their own interest and were not known for Equation (1.4) and the abstract version (1.5). For example, to define the global manifold structure of the stable and unstable sets of equilibria (see [22]), we need to prove that (1.5), as well as the linearized equation (and its adjoint) satisfy a backward uniqueness property. We will also prove that Equation (1.5) generates an asymptotically smooth dynamical system. This property ensures the compactness of any bounded set of trajectories. Besides, we use the asymptotic smoothness to prove that the globally bounded solutions of (1.5) are analytic in time, when the nonlinearity f is analytic in u . This property plays a central role in the last part of the proof of the abstract theorem 2.6. In this paper, we also show how to use the structure of the spectrum of the linear operator A to obtain all these auxiliary results. Indeed, the fact that the set of rootvectors of A is a Riesz basis of the functional space enables simple and elegant proofs of such qualitative properties.

To prove the genericity of Kupka-Smale property, we follow the lines of [6]. The main new difficulties appear at the end of the proof, when we must prove that the integral introduced in [5] and [6] (see (5.8) in our case) does not always vanish. Indeed, this step strongly uses the properties of the spectrum, and so had to be modified. Moreover, the development of the bounded solutions ψ of the adjoint equation in the neighborhood of $t = -\infty$ is more involved (see Proposition 5.5 and Lemma 5.8). In order to estimate the decay of the real function $\psi(x_0, t)$ at some point $x_0 \in \Omega$, Brunovský and Raugel used a property of almost-periodicity coming from the fact that the non-real eigenvalues of the operator all lay on a same vertical line. In the non-constant damping case, the spectrum is more complicated and such argument cannot be applied. We replaced it by a Laplace transform argument.

Section 2 of this paper is devoted to the statement of the abstract theorem (Theorem 2.6) and the proof of Theorem 1.1. In Section 3, we deal with the case of Equation (1.4) and introduce another example. We also discuss which properties still hold in cases where the hypotheses of Theorem 2.6 are only partially satisfied. The proof of our main theorem is split into two parts. The first part deals with generic properties of the spectrum, including hyperbolicity or simplicity of the eigenvalues. These properties, given in Section 4, are worth a separated section, as they have their own interest. The second part of the proof of our main theorem is given in Section 5. It shows how to apply the Brunovský-Poláčik-Raugel theorem A.1.

Acknowledgements :

The author wants to thank Geneviève Raugel for her patience and all her advices and remarks, without which this work could not have been done. The author also thanks Patrick Gérard for fruitful discussions and the referee for his valuable comments.

2 Abstract genericity theorem

In this section, we are going to state our abstract theorem. We first introduce the frame in which we will work. In particular, we describe the space X and the operators A , B and Γ that we introduced in the introduction.

We want our theorem to be as simple as possible and, at the same time, to be directly applicable to as many situations as possible. For these reasons, our assumptions on A will be as basic as possible and thus we will have to do some preliminaries to be able to state our main Theorem 2.6.

In order to make the reading easier, we illustrate each abstract hypothesis with the corresponding property in the case of the internal damped wave equation (1.3). As a result, Theorem 1.1 will be a direct corollary of Theorem 2.6. We briefly recall that the damped wave equation (1.3) and its analogue in higher dimension have been extensively studied for a long time ([11], [35], [13], [38]...). In the one-dimensional case, the exponential decay of the linear semigroup has been proved in [21] (see also [8] and [11]). In higher dimensions, the exponential decay is still true under additional conditions ([2], [35] and [38]). In these cases, the regularity of the complete bounded orbits is proved in [20].

2.1 Introduction of the abstract wave equation

We work in $\mathbb{L}^2(]0, 1[)$ with the usual scalar product

$$\langle u|v \rangle_{\mathbb{L}^2} = \int_0^1 u(x)\overline{v(x)}dx .$$

In order to define the operator A , we introduce the operators B and Γ , which satisfy the following hypotheses.

- (B) B is a real positive self-adjoint operator from its domain $D(B)$ into $\mathbb{L}^2(]0, 1[)$. Moreover, we assume that $B^{-1/2}$ is smoothing in the sense that $B^{-1/2}$ defines a continuous linear mapping from $\mathbb{H}^\alpha(]0, 1[)$ into $\mathbb{H}^{\alpha+1}(]0, 1[)$ for all $\alpha \geq 0$ (in particular $D(B^{1/2})$ is continuously imbedded in $\mathbb{H}^1(]0, 1[)$).

(Gam) Γ is a continuous linear operator from $D(B^{1/2})$ into $D(B^{1/2})$. In addition, Γ is a compact nonnegative self-adjoint operator on $D(B^{1/2})$. In particular, for any φ and ψ in $D(B^{1/2})$, we have

$$\langle B^{1/2}\Gamma\varphi|B^{1/2}\psi \rangle_{\mathbb{L}^2} = \langle B^{1/2}\varphi|B^{1/2}\Gamma\psi \rangle_{\mathbb{L}^2} .$$

We introduce the Banach space $X = D(B^{1/2}) \times \mathbb{L}^2$ endowed with the natural scalar product :

$$\begin{aligned} \left\langle \begin{pmatrix} u \\ v \end{pmatrix} \middle| \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\rangle_X &= \langle B^{1/2}u|B^{1/2}\varphi \rangle_{\mathbb{L}^2} + \langle v|\psi \rangle_{\mathbb{L}^2} \\ &:= \langle u|\varphi \rangle_{D(B^{1/2})} + \langle v|\psi \rangle_{\mathbb{L}^2} \end{aligned} \quad (2.1)$$

Let A be the operator

$$A : D(A) \longrightarrow X , \quad A \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -B(u + \Gamma v) \end{pmatrix} , \quad (2.2)$$

where $D(A)$ is defined as follows

$$D(A) = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in X \middle/ v \in D(B^{1/2}), (u + \Gamma v) \in D(B) \right\} .$$

Example of Equation (1.3) : We set $B = -\Delta_D$, where Δ_D is the Laplacian with homogeneous Dirichlet boundary condition. We have

$$D(B) = \mathbb{H}^2(]0, 1[) \cap \mathbb{H}_0^1(]0, 1[) \text{ and } X = \mathbb{H}_0^1(]0, 1[) \times \mathbb{L}^2(]0, 1[) .$$

The operator Γ is defined as follows.

$$\Gamma : \begin{pmatrix} \mathbb{H}_0^1(]0, 1[) & \longrightarrow & \mathbb{H}_0^1(]0, 1[) \\ v & \longmapsto & B^{-1}(\gamma(x)v) \end{pmatrix} .$$

The operator Γ is compact. Moreover it is nonnegative and self-adjoint on $\mathbb{H}_0^1(]0, 1[)$, since for any $(v, v') \in \mathbb{H}_0^1(]0, 1[)^2$, we have

$$\langle \Gamma v|v' \rangle_{D(B^{1/2})} = \int_0^1 \gamma(x)v(x)\overline{v'(x)} dx .$$

The operator A is simply the classical damped wave operator

$$A = \begin{pmatrix} 0 & Id \\ \Delta_D & -\gamma(x) \end{pmatrix} .$$

Remark : Actually, we do not need to assume that B is positive. It suffices to suppose that there exists a positive number λ such that $B' = B + \lambda Id$ satisfies the property (B). An example is $B = -\Delta_N$, where Δ_N is the Laplacian with Neumann boundary conditions. In such cases, we can work with B' by replacing $f(x, u)$ by $f(x, u) - \lambda u$ in Equation (2.4).

The hypotheses (B) and (Gam) imply directly the following lemma.

Lemma 2.1. *For any function $h \in \mathbb{L}^\infty([0, 1], \mathbb{C})$ and any complex number λ , the operator L from $D(B^{1/2})$ into $D(B^{1/2})$ defined by*

$$L\varphi = \varphi + \lambda\Gamma\varphi - B^{-1}(h\varphi),$$

is a Fredholm operator of index 0 and $L^ = \bar{L}$ on $D(B^{1/2})$. In particular, g is in the range of L if and only if, for all φ in the kernel of L , $\langle g|\bar{\varphi} \rangle_{D(B^{1/2})} = 0$.*

Proof : The operator $B^{-1}(h.)$ is defined from $D(B^{1/2})$ into $D(B)$, so it is compact from $D(B^{1/2})$ into $D(B^{1/2})$. By assumption, Γ is also compact on $D(B^{1/2})$, so, as L is a compact perturbation of the identity on $D(B^{1/2})$, L is a Fredholm operator of index 0 (see for example [3]). $L^* = \bar{L}$, indeed if v and φ are two functions of $D(B^{1/2})$, we have

$$\begin{aligned} \langle L\varphi|v \rangle_{D(B^{1/2})} &= \langle \varphi|v \rangle_{D(B^{1/2})} + \lambda \langle \Gamma\varphi|v \rangle_{D(B^{1/2})} - \langle B^{-1/2}(h\varphi)|B^{1/2}v \rangle_{\mathbb{L}^2} \\ &= \langle \varphi|v \rangle_{D(B^{1/2})} + \lambda \langle \varphi|\Gamma v \rangle_{D(B^{1/2})} - \langle B^{1/2}\varphi|B^{-1/2}(\bar{h}v) \rangle_{\mathbb{L}^2} \\ &= \langle \varphi|v + \bar{\lambda}\Gamma v - B^{-1}(\bar{h}v) \rangle_{D(B^{1/2})} \\ &= \langle \varphi|\bar{L}v \rangle_{D(B^{1/2})} \end{aligned}$$

The last claim of the lemma is just the Fredholm alternative. □

As a consequence of Lemma 2.1, we can prove that A generates a \mathcal{C}^0 -semigroup.

Proposition 2.2. *A is a nonpositive operator and $Id - A$ is surjective from $D(A)$ onto X . As a consequence, A generates a \mathcal{C}^0 -semigroup e^{At} of contractions on X . Moreover, A has a compact resolvent.*

Proof : If $(u, v) \in D(A)$, then

$$\left\langle A \begin{pmatrix} u \\ v \end{pmatrix} \middle| \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle_X = - \langle \Gamma v|v \rangle_{D(B^{1/2})} ,$$

so A is nonpositive and thus dissipative. We next prove the maximality, that is that $Id - A$ is surjective. If $(h, g) \in X$, there exists a vector $(u, v) \in D(A)$ such that

$$(Id - A) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} h \\ g \end{pmatrix}$$

if and only if

$$\begin{cases} u - v = h \\ v + B(u + \Gamma v) = g \end{cases} \quad (2.3)$$

This is equivalent to find a function $u \in D(B^{1/2})$ such that

$$(u - h) + \Gamma(u - h) + B^{-1}(u - h) = B^{-1}g - h.$$

Lemma 2.1 implies that this is possible if $Ker(Id + \Gamma + B^{-1}) = \{0\}$. But if $\varphi \in D(B^{1/2})$ is such that

$$\varphi + \Gamma\varphi + B^{-1}\varphi = 0,$$

then

$$\|\varphi\|_{D(B^{1/2})}^2 + \langle \Gamma\varphi | \varphi \rangle_{D(B^{1/2})} + \|\varphi\|_{\mathbb{L}^2}^2 = 0,$$

and since $\langle \Gamma\varphi | \varphi \rangle_{D(B^{1/2})} \geq 0$, it follows that $\varphi = 0$. As a consequence of the Lumer-Phillips Theorem (see [33]), A generates a \mathcal{C}^0 -semigroup e^{At} on X .

Finally, we can easily prove that $(Id - A)^{-1}$ is compact, using the equalities (2.3), and the fact that B^{-1} and Γ are compact on $D(B^{1/2})$. \square

Let f be any function of $\mathcal{C}^k([0, 1] \times \mathbb{R})$, $k \geq 1$. Since the map

$$\begin{pmatrix} u \\ v \end{pmatrix} \in X \longmapsto \begin{pmatrix} 0 \\ f(x, u) \end{pmatrix} \in X$$

is of class \mathcal{C}^1 , the equation

$$\frac{\partial}{\partial t} \begin{pmatrix} u \\ u_t \end{pmatrix} = A \begin{pmatrix} u \\ u_t \end{pmatrix} + \begin{pmatrix} 0 \\ f(x, u) \end{pmatrix} \quad t > 0, \quad \begin{pmatrix} u \\ u_t \end{pmatrix} \Big|_{t=0} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in X \quad (2.4)$$

has a unique local solution $(u, u_t) \in \mathcal{C}^0([0, T[, X)$ where T is a positive time, which depends on the initial data (u_0, u_1) . We denote by $S(t)(u_0, u_1) = (u, u_t)$ the solution in $\mathcal{C}^0([0, T[, X)$ of (2.4) and remark that $S(t)$ is a local nonlinear semigroup on X . If (u, u_t) is a solution in $\mathcal{C}^0([0, T[, X)$ of Equation (2.4), the linearized equation along the solution (u, u_t) is given by

$$\frac{\partial}{\partial t} \begin{pmatrix} w \\ w_t \end{pmatrix} = A \begin{pmatrix} w \\ w_t \end{pmatrix} + \begin{pmatrix} 0 \\ f'_u(x, u(x, t))w \end{pmatrix}, t > 0, \quad \begin{pmatrix} w \\ w_t \end{pmatrix} \Big|_{t=0} = \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} \in X. \quad (2.5)$$

Now, we will look at the adjoint operator A^* . First notice that, with our choice of the scalar product, we have $X^* = X$. Due to Hypothesis (Gam), we verify that

$$A^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.6)$$

Notice that (2.6) means that the adjoint linear equation $\frac{\partial}{\partial t}(u, u_t) = -A^*(u, u_t)$ is “equivalent” to the equation $\frac{\partial}{\partial t}(u, u_t) = A(u, u_t)$ where the time is reversed. Finally, the adjoint equation of (2.4) is given by

$$\begin{cases} \frac{\partial}{\partial t} \begin{pmatrix} \theta \\ \psi \end{pmatrix} = -A^* \begin{pmatrix} \theta \\ \psi \end{pmatrix} - \begin{pmatrix} B^{-1}(f'_u(x, u)\psi) \\ 0 \end{pmatrix} & t < 0 \\ \begin{pmatrix} \theta \\ \psi \end{pmatrix}_{t=0} = \begin{pmatrix} \theta_0 \\ \psi_0 \end{pmatrix} \in X \end{cases} \quad (2.7)$$

To obtain our main theorem, we need to introduce spectral hypotheses.

2.2 Spectral assumptions

As A has a compact resolvent, its spectrum consists only of isolated eigenvalues λ_n with finite multiplicity. With each λ_n , we can associate an orthonormalized Jordan Chain

$$(V_{j,k}^n)_{j \leq m_n - 1, k \leq m_{n,j} - 1}$$

with

$$\begin{aligned} \forall 0 \leq j \leq m_n - 1, \quad AV_{j,0}^n &= \lambda_n V_{j,0}^n \\ \forall 0 \leq j \leq m_n - 1, 1 \leq k \leq m_{n,j} - 1, \quad AV_{j,k}^n &= \lambda_n V_{j,k}^n + V_{j,k-1}^n \end{aligned}$$

We will assume that the rootvectors form a Riesz basis of X . We say that a set $(\Psi_n)_{n \in \mathbb{N}}$ is a Riesz basis of X if there exist two positive constants a_1 and a_2 , such that for any $U \in X$, there is a unique sequence of complex numbers (α_n) such that

$$U = \sum_{n \in \mathbb{N}} \alpha_n \Psi_n,$$

and

$$a_1 \|U\|_X^2 \leq \sum_{n \in \mathbb{N}} |\alpha_n|^2 \leq a_2 \|U\|_X^2. \quad (2.8)$$

See [14] for details.

We assume the following spectral properties

(Spec) the operator A is such that

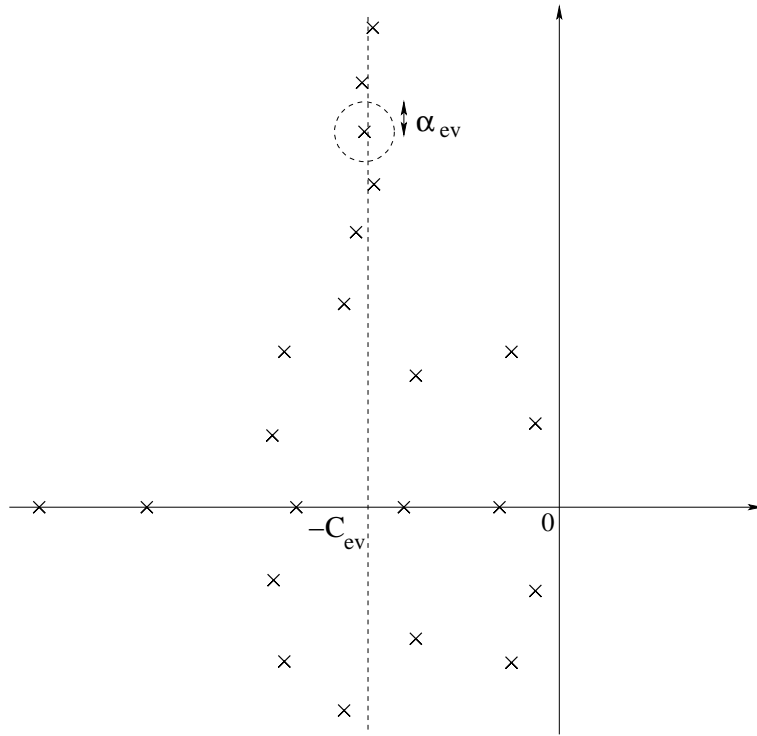
- a) the rootvectors $(V_{j,k}^n)$ of A form a Riesz basis of X .
- b) there exist two positive constants M_{ev} and C_{ev} such that all eigenvalues λ with $|\lambda| > M_{ev}$ are simple and can be written as

$$\lambda_{\pm n} = a_n \pm ib_n \quad \text{with} \quad a_n \longrightarrow -C_{ev} . \quad (2.9)$$

- c) the eigenvalues are uniformly isolated in the sense that there exists a constant $\alpha_{ev} > 0$ such that

$$\inf_{n \neq m} |\lambda_n - \lambda_m| > \alpha_{ev} .$$

In short, the spectrum looks like



Example of Equation (1.3) : It has been proved by Cox and Zuazua in [11] that the spectrum of Equation (1.3) satisfies Hypothesis (Spec). In particular, they showed the following high-frequency estimate

$$\lambda_{\pm n} = -\frac{\gamma_0}{2} \pm in\pi + O\left(\frac{1}{n}\right) ,$$

where γ_0 is the average of γ on $[0, 1]$, that is,

$$\gamma_0 = \int_0^1 \gamma(x) dx > 0 .$$

We will use an important property which easily follows from Hypothesis (Spec), but has to be enhanced.

Proposition 2.3. *Under Hypothesis (Spec), the eigenvalues of A satisfy*

$$\sum_{\lambda \in \sigma(A)} \frac{1}{|\lambda|^2} < \infty.$$

We assume furthermore that

(UCP) B and Γ satisfy the following weak unique continuation property : for any given $\lambda \in \mathbb{C}$, any function $h \in \mathbb{L}^\infty$, and any $\varphi \in D(B^{1/2}) \setminus \{0\}$ with $(\varphi + \lambda\Gamma\varphi) \in D(B)$ which satisfy

$$B(\varphi + \lambda\Gamma\varphi) = h(x)\varphi ,$$

a) φ vanishes only at a finite number of points of $[0, 1]$.

b) $\langle \Gamma\varphi | \varphi \rangle_{D(B^{1/2})} > 0$.

Example of Equation (1.3) : *Notice that Property (UCP)a) for Equation (1.3) corresponds to the following well-known fact. If $\varphi \in \mathbb{H}_0^1(]0, 1[) \setminus \{0\}$ satisfies*

$$\varphi_{xx}(x) = (\lambda\gamma(x) - h(x))\varphi(x) ,$$

then φ vanishes only at a finite number of points of $[0, 1]$. In particular,

$$\langle \Gamma\varphi | \varphi \rangle_{D(B^{1/2})} = \int_0^1 \gamma(x)|\varphi(x)|^2 > 0 ,$$

and thus Hypothesis (UCP)b) obviously holds.

From the assumptions (Spec) and (UCP), we deduce the exponential decay property of the linear semigroup e^{At} . We deduce as well that the semigroup e^{At} is in fact a group.

Proposition 2.4. *Under the assumptions (B), (Gam), (Spec) and (UCP), the linear semigroup e^{At} is decreasing with an exponential decay rate. More precisely, there exist $\delta_d^1 > \delta_d^2 > 0$ and two positive constants K_d^1 and K_d^2 , such that for any $U_0 \in X$ and $t > 0$,*

$$K_d^1 e^{-\delta_d^1 t} \|U_0\|_X \leq \|e^{At}U_0\|_X \leq K_d^2 e^{-\delta_d^2 t} \|U_0\|_X . \quad (2.10)$$

Besides, the \mathcal{C}^0 -semigroup $(e^{At})_{t \in \mathbb{R}_+}$ can be extended to a \mathcal{C}^0 -group $(e^{At})_{t \in \mathbb{R}}$. In the same way, the wave equation (2.4), the linearized equation (2.5) and the adjoint equation (2.7) generate \mathcal{C}^0 -groups and thus satisfy the backward uniqueness property.

Proof : Since we assumed that there exists a Riesz basis $(V_{j,k}^n)$ composed by rootvectors of A , we can write

$$U_0 = \sum_{n,j,k} \alpha_{n,j,k} V_{j,k}^n ,$$

where the series converges normally in X . As

$$e^{tA} V_{j,k}^n = (V_{j,k}^n + t V_{j,k-1}^n + \dots + t^k V_{j,0}^n) e^{\lambda_n t} ,$$

we have that

$$e^{tA} U_0 = \sum_{n,j,k} \alpha_{n,j,k} (V_{j,k}^n + t V_{j,k-1}^n + \dots + t^k V_{j,0}^n) e^{\lambda_n t} .$$

By assumption (Spec)b), there is only a finite number of eigenvalues which are not simple, so the multiplicity of an eigenvalue is uniformly bounded and the polynomial terms do not matter. Since we have the equivalence of norms (2.8), to conclude, we have to show that there exist $\delta_d^1 > \delta_d^2 > 0$ such that all the eigenvalues λ_n of A are in the strip

$$-\delta_d^1 < \operatorname{Re}(\lambda_n) < -\delta_d^2 < 0 .$$

As Property (Spec)b) holds, it is sufficient to prove that all the eigenvalues have negative real part. We proved in Proposition 2.2 that A is nonpositive and so its eigenvalues have nonpositive real part. If $i\lambda \in i\mathbb{R}$ is an eigenvalue of A with eigenvector $(\varphi, i\lambda\varphi)$ ($\varphi \neq 0$), then

$$\varphi + i\lambda\Gamma\varphi - \lambda^2 B^{-1}\varphi = 0 ,$$

and

$$\|\varphi\|_{D(B^{1/2})}^2 + i\lambda \langle \Gamma\varphi | \varphi \rangle_{D(B^{1/2})} - \lambda^2 \|\varphi\|_{\mathbb{L}^2}^2 = 0 .$$

Due to the assumption (UCP)b), $\langle \Gamma\varphi | \varphi \rangle_{D(B^{1/2})} \neq 0$, which implies that $\lambda = 0$, and thus $\varphi = 0$. This proves that all the eigenvalues have negative real part.

To show that e^{At} can be extended to a group, we formally define e^{-At} . Let

$$U_0 = \sum_{n,j,k} \alpha_{n,j,k} V_{j,k}^n .$$

We denote by $\beta_{n,j,0}^t, \dots, \beta_{n,j,m_{n,j}-1}^t$, the solutions of the system

$$\begin{pmatrix} 1 & t & \dots & t^{m_{n,j}-1} \\ 0 & 1 & \dots & t^{m_{n,j}-2} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_{n,j,0}^t \\ \beta_{n,j,1}^t \\ \vdots \\ \beta_{n,j,m_{n,j}-1}^t \end{pmatrix} = \begin{pmatrix} \alpha_{n,j,0} \\ \alpha_{n,j,1} \\ \vdots \\ \alpha_{n,j,m_{n,j}-1} \end{pmatrix} .$$

Notice that the above system is well-defined for all time $t \geq 0$. We set

$$e^{-At} U_0 = \sum_n e^{-\lambda_n t} \left(\sum_{j,k} \beta_{n,j,k}^t V_{j,k}^n \right) .$$

The property (2.8), together with the fact that the real part and the multiplicity of the eigenvalues are bounded, imply that the operator e^{-At} is linear continuous from X into X , and it continuously depends on t . Besides, $e^{-At} = (e^{At})^{-1}$ by construction. This shows that e^{At} is a group (see [33]). \square

The following proposition shows that the exponential decay of e^{At} implies that the dynamical system $S(t)$ generated by Equation (2.4) is asymptotically smooth. We recall that it means that any bounded positively invariant set \mathcal{B} of X is attracted by a compact set $K(\mathcal{B}) \subset \overline{\mathcal{B}}$ (see [17]).

Proposition 2.5. *The dynamical system $S(t)$ is asymptotically smooth.*

Proof : Let f be a function of \mathfrak{G}^k and (u_0, v_0) be initial data in X . We denote $(u(t), v(t))$ the trajectory $S(t)(u_0, v_0)$. By definition of a mild solution, we have

$$S(t) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = e^{tA} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \int_0^t e^{(t-s)A} \begin{pmatrix} 0 \\ f(x, u(s)) \end{pmatrix} ds.$$

The linear term satisfies $\|e^{tA}\|_{\mathcal{L}(X)} \leq K_d^2 e^{-\delta_a^2 t}$. Moreover, the mapping $F : (u, v) \mapsto (0, f(x, u))$, defined from X to X , is completely continuous in the sense of [17]. Indeed, if (u, v) belongs to a bounded set $\mathcal{B} \subset X$ then $f(x, u)$ belongs to a bounded set of $\mathbb{H}^1(]0, 1[)$ and thus $F(\mathcal{B})$ is a precompact set of X . Applying Theorem 4.6.1 of [17] yields that $S(t)$ is asymptotically smooth. \square

2.3 The main theorem

As the backward uniqueness property is satisfied by the linearized equation (2.5) and the adjoint equation (2.7), the stable and unstable sets of the hyperbolic equilibria of the equation (2.4) are imbedded manifolds in X (see [22]), so wondering if their intersection is transversal or not has a sense. We recall that (2.4) is said to have the Kupka-Smale property, if all its equilibria are hyperbolic and all its stable and unstable manifolds intersect transversally. We also recall that for $k \geq 1$, \mathfrak{G}^k is the set of the functions $f \in \mathcal{C}^k([0, 1] \times \mathbb{R}, \mathbb{R})$ endowed with the Whitney topology, that is, the topology generated by the sets

$$\{g \in \mathfrak{G}^k / |D^i f(x, u) - D^i g(x, u)| < \delta(u), i = 0, 1, \dots, k, x \in [0, 1], u \in \mathbb{R}\},$$

where δ is any positive function on \mathbb{R} .

We are now able to state our main theorem.

Theorem 2.6. *Let f be a function of \mathfrak{G}^k . We assume that the operator A defined by (2.2) satisfies all the above properties (B), (Gam), (Spec) and (UCP). We assume in addition that the following properties hold :*

(Grad) *The Equation (2.4) is a gradient system.*

(Loc) *For any function $\varphi \in D(B^{1/2})$, the scalar product $\langle \Gamma\varphi | \varphi \rangle_{D(B^{1/2})}$ depends only on the values of φ^2 .*

If $k \geq 2$, then the set \mathfrak{G}^{KS} of all the functions $f \in \mathfrak{G}^k$ such that (2.4) has the Kupka-Smale property is a generic subset of \mathfrak{G}^k .

If f satisfies an additional dissipative condition, Equation (2.4) has a compact global attractor. In this case, if (2.4) is Kupka-Smale, then it is Morse-Smale ; that is, (2.4) has a finite number of equilibria which are all hyperbolic and all its stable and unstable manifolds intersect transversally. Let $\mathfrak{G}_{\text{diss}}^k$ be the open subset of \mathfrak{G}^k defined by

$$\mathfrak{G}_{\text{diss}}^k = \{f \in \mathfrak{G}^k / \limsup_{u \rightarrow \pm\infty} \frac{f(x, u)}{u} < 0\} .$$

When f belongs to $\mathfrak{G}_{\text{diss}}^k$, the above theorem can be improved as follows.

Corollary 2.7. *We assume that all the hypotheses of Theorem 2.6 are satisfied. If f belongs to $\mathfrak{G}_{\text{diss}}^k$, then Equation (2.4) admits a compact global attractor. As a consequence, the set of non-linearities $f \in \mathfrak{G}_{\text{diss}}^k$ such that (2.4) has the Morse-Smale property is generic in $\mathfrak{G}_{\text{diss}}^k$.*

Proof : We only have to check that, if f belongs to $\mathfrak{G}_{\text{diss}}^k$, then Equation (2.4) admits a compact global attractor. We follow the proof given in [19]. We have already shown that (2.4) generates an asymptotically smooth system $S(t)$. Next, we have to prove that $S(t)$ is point-dissipative, that is that there exists a bounded set which attracts each point of X . As f belongs to $\mathfrak{G}_{\text{diss}}^k$, there exist two positive constants α and C such that

$$f(x, u)u \leq C - \alpha u^2 .$$

This implies that the equilibrium points $(e, 0)$ of Equation (2.4) are uniformly bounded, since

$$\|(e, 0)\|_X^2 = \|e\|_{D(B^{1/2})}^2 = \int (Be)(x)e(x)dx = \int f(x, e(x))e(x)dx \leq C .$$

The asymptotically smooth system $S(t)$ is assumed to be gradient, so all the points of X are attracted by the equilibria, which belong to a bounded set. Thus, $S(t)$ is point-dissipative.

Finally, we have to verify that the trajectories of the bounded sets of X are bounded. More precisely, let \mathcal{B} be a bounded set of X , we set $\gamma_+(\mathcal{B}) = \cup_{t \geq 0} S(t)\mathcal{B}$. To obtain the existence of a compact global attractor, it remains to check that $\gamma_+(\mathcal{B})$ is bounded. We define the functional Φ from X into \mathbb{R} as follows.

$$\Phi((u_0, v_0)) = \frac{1}{2} \|(u_0, v_0)\|_X^2 - \int_0^1 F(x, (u_0, v_0)) dx ,$$

where

$$F(x, (u_0, v_0)) = \int_0^{u_0} f(x, \zeta) d\zeta .$$

If $(u_0, v_0) \in D(A)$, then $(u, v)(t) = S(t)(u_0, v_0)$ belongs to $\mathcal{C}^0(\mathbb{R}_+, D(A)) \cap \mathcal{C}^1(\mathbb{R}_+, X)$, and Φ is non-increasing along this trajectory since we can write

$$\frac{\partial}{\partial t} \Phi((u, v)(t)) = - \langle \Gamma v | v \rangle_{D(B^{1/2})} \leq 0 .$$

As $S(t)$ and Φ are continuous on X , and that $D(A)$ is dense in X , we deduce that Φ is non-increasing along all the trajectories of $S(t)$. In fact, Φ is a strict Lyapunov function in the concrete examples. The first components u_0 of the elements (u_0, v_0) of \mathcal{B} are uniformly bounded in $\mathbb{H}^1(]0, 1[)$, so Φ is also bounded on \mathcal{B} by a constant $C(\mathcal{B})$. As f belongs to $\mathfrak{G}_{\text{diss}}^k$, there exists two positive constants $\tilde{\alpha}$ and \tilde{C} such that

$$F(x, u) \leq \tilde{C} - \tilde{\alpha} u^2 .$$

This implies that

$$\frac{1}{2} \|(u, v)\|_X^2 - \tilde{C} \leq \Phi((u, v)) \leq \Phi((u_0, v_0)) \leq C(\mathcal{B}) .$$

This shows that $\gamma_+(\mathcal{B})$ is bounded in X . □

Proof of Theorem 1.1 and Corollary 1.2 : As already indicated, Theorem 1.1 and Corollary 1.2 are direct consequences of Theorem 2.6 and Corollary 2.7. While introducing the hypotheses (B), (Gam), (Spec) and (UCP), we have checked that Equation (1.3) satisfies all these conditions. Moreover, it is known that (1.3) is a gradient system (see for example [19]). In addition, the property (Loc) is obviously satisfied as

$$\forall \varphi \in D(B^{1/2}), \quad \langle \Gamma \varphi | \varphi \rangle_{D(B^{1/2})} = \int_0^1 \gamma(x) |\varphi(x)|^2 dx .$$

So we can apply Theorem 2.6 and Corollary 2.7. □

3 Other examples of applications

In this section, we apply the abstract theorem to the case of the boundary damping and thus prove Theorem 1.3. We give also another example, which illustrates the case of operators of higher order and which is interesting in the sense that we will need to generalize Theorem 2.6. Of course, these examples are not exhaustive. In particular, other boundary conditions can be taken in Equation (1.3). We also notice that Hypothesis (Spec) has been proved for many other one-dimensional equations.

In the last subsection, we enhance which properties are still true for equations which do not satisfy all the hypotheses of Theorem 2.6.

3.1 The wave equation with damping on the boundary

Like the internal damped wave equation (1.3), the equation with boundary damping (1.4) has also attracted much attention (see for example [9], [12], [26], [27], [28], [37] and [39]). Theorem 1.3 is a direct consequence of Theorem 2.6 and Corollary 2.7.

Proof of Theorem 1.3 : We can assume without loss of generality that $\gamma(0) \neq 0$ (the case $\gamma(0) = 0$ and $\gamma(1) \neq 0$ is similar). Equation (1.4) can be written in the frame of (2.4) with A defined by (2.2). Indeed, we set

$$X = \mathbb{H}^1(]0, 1[) \times \mathbb{L}^2(]0, 1[), \quad B = -\partial_{xx}^2 + Id, \quad D(B) = \{u \in \mathbb{H}^2(]0, 1[) / u_x(0) = u_x(1) = 0\},$$

and, for any $v \in \mathbb{H}^1(]0, 1[)$, we denote by Γv the solution in $\mathbb{H}^2(]0, 1[)$ of

$$\begin{cases} (\partial_{xx}^2 - Id)(\Gamma v)(x) = 0 & x \in]0, 1[\\ \frac{\partial}{\partial v}(\Gamma v)(x) = \gamma(x)v(x) & x \in \{0, 1\} \end{cases} .$$

We recall that we equip the space $X = D(B^{1/2}) \times \mathbb{L}^2(]0, 1[)$ with the inner product defined by (2.1). For any $v \in \mathbb{H}^1(]0, 1[)$, we have

$$\langle \Gamma v | v \rangle_{D(B^{1/2})} = \gamma(0)|v(0)|^2 + \gamma(1)|v(1)|^2 . \quad (3.1)$$

Thus, the operators B and Γ clearly satisfy Hypotheses (B), (Gam) and (Loc). Following Cox and Zuazua (see [12]), we prove that Hypothesis (Spec) holds. Let

$$\omega = \left(\frac{\gamma(0) + 1}{\gamma(0) - 1} \right) \left(\frac{\gamma(1) + 1}{\gamma(1) - 1} \right) .$$

We have the following high-frequency estimate

$$\lambda_{\pm n} = -\ln |\omega| + \begin{cases} \pm in\pi + O(\frac{1}{n}) & \text{if } \omega > 0 \\ \pm i(n + \frac{1}{2})\pi + O(\frac{1}{n}) & \text{if } \omega < 0 \end{cases} .$$

Hypothesis (UCP)a) is a well-known unique continuation property (see [31]). To show the property (UCP)b), we assume that $\lambda \in \mathbb{C}$, $h \in \mathbb{L}^\infty$, and $\varphi \neq 0$ is such that $(\varphi + \lambda\Gamma\varphi) \in D(B)$ and

$$\begin{cases} B(\varphi + \lambda\Gamma\varphi) = h(x)\varphi \\ \langle \Gamma\varphi | \varphi \rangle_{D(B^{1/2})} = 0 \end{cases} \quad \text{that is} \quad \begin{cases} -\varphi_{xx}(x) = h(x)\varphi(x), & x \in]0, 1[\\ \frac{\partial\varphi}{\partial\nu}(x) = -\lambda\gamma(x)\varphi(x), & x = 0, 1 \\ \gamma(0)|\varphi(0)|^2 + \gamma(1)|\varphi(1)|^2 = 0 \end{cases} .$$

Thus, φ satisfies both Neumann and Dirichlet conditions at the end point $x=0$. Simply using Cauchy-Lipschitz Theorem, we find that $\varphi = 0$, that is that Hypothesis (UCP)b) is satisfied.

Finally, it remains to show that (2.4) generates a gradient system. Although the proof is classical (see [27]), we quickly recall it. The Lyapounov function associated to (2.4) is given by

$$\Phi((u_0, v_0)) = \frac{1}{2} \|(u_0, v_0)\|_X^2 - \int_0^1 F(x, (u_0, v_0)) dx ,$$

where

$$F(x, (u_0, v_0)) = \int_0^{u_0} f(x, \zeta) d\zeta .$$

Along a trajectory $(u, v) = S(t)(u_0, v_0)$ of Equation (1.4), we have

$$\frac{\partial}{\partial t} \Phi((u, v)(t)) = - \langle \Gamma v | v \rangle_{D(B^{1/2})} \leq 0 .$$

Moreover, if the trajectory is such that $\frac{\partial}{\partial t} \Phi((u, v)(t)) = 0$ for $t \in [0, T]$, then v satisfies $v(0) = \frac{\partial}{\partial x} v(0) = 0$. We set $v_0(x) = v(x, 0)$ and $v_T(x) = v(x, T)$. If we reverse the role of time and space, v is a solution of

$$\begin{cases} \frac{\partial^2}{\partial x^2} v(x, t) = (\frac{\partial^2}{\partial t^2} + Id - f'_u(x, u(x, t)))v(x, t) & 0 < x < 1, t \in]0, T[\\ v(x, 0) = v_0(x), v(x, T) = v_T(x) & 0 < x < 1 \\ v(x = 0, t) = \frac{\partial}{\partial x} v(x = 0, t) = 0 & t \in]0, T[. \end{cases}$$

The uniqueness of the solutions of the wave equation gives that $v(x, t) = u_t(x, t) = 0$ for $x \in]0, 1[$ and $t \in]0, T[$. This implies that Φ is a strict Lyapounov function, and so that (2.4) generates a gradient system. \square

Remarks :

- When $\gamma(0) = 1$ or $\gamma(1) = 1$, the spectrum of the linear operator defined in Equation (1.4) by setting $f = 0$ is empty. Moreover, it is well-known that, in this case, all solutions of the linear equation vanish in finite time, that is that Equation (1.4) does not satisfy the backward uniqueness property. Thus, we cannot ensure that the stable and

unstable manifolds are immersed manifolds as they may have self-intersections. In this case, we are not able even to define the notion of transversality of these manifolds.

- In [12], Cox and Zuazua considered Equation (1.4) with $\rho(x)^2 u_{tt}$ instead of u_{tt} , where ρ is a measurable function with bounded variation and satisfies $0 < \alpha \leq \rho \leq \beta < \infty$. We can apply Theorem 2.6 to this case if we notice that our theorem is also valid when $\mathbb{L}^2(]0, 1[)$ is replaced by $\mathbb{L}_\rho^2(]0, 1[)$, that is the space \mathbb{L}^2 endowed with the equivalent norm $\|f\|_\rho^2 = \int \rho^2 |f|^2$.

3.2 A beam equation with joint feedback control

In Hypothesis (Spec)b), we assumed that there exists a constant C_{ev} such that the eigenvalues of A satisfy

$$\operatorname{Re}\lambda_n \longrightarrow -C_{ev} .$$

Obviously, a careful look at the proof of Theorem 2.6 shows that it is also valid if we assume that the sequence $(\operatorname{Re}(\lambda_n))_{n \in \mathbb{Z}}$ has a finite number of accumulation points. The following example, described in [16], illustrates this slight generalization.

Let f be a function in \mathfrak{G}^k , γ and K be two positive constants and let d be a point of $]0, 1[$. We study in

$$X = (\mathbb{H}^2(]0, 1[) \cap \mathbb{H}_0^1(]0, 1[)) \times \mathbb{L}^2(]0, 1[)$$

the equation

$$\left\{ \begin{array}{ll} u_{tt}(x, t) + \gamma u_t(x, t) = -u_{xxxx}(x, t) + f(x, u) & 0 < x < d, d < x < 1, t > 0 \\ u(x, t) = u_{xx}(x, t) = 0 & x = 0, 1, t > 0 \\ \partial_x^k u(d_+, t) = \partial_x^k u(d_-, t) & k = 0, 1, 2, t > 0 \\ u_{xxx}(d_-, t) - u_{xxx}(d_+, t) = K u_t(d, t) & t > 0 \\ (u(x, 0), u_t(x, 0)) = (u_0(x), u_1(x)) \in X & x \in [0, 1] \end{array} \right. \quad (3.2)$$

where, if h is a piecewise continuous function, we denote

$$h(d^+) = \lim_{x \rightarrow d, x > d} h(x) \quad \text{and} \quad h(d^-) = \lim_{x \rightarrow d, x < d} h(x) .$$

Theorem 3.1. *If d is a rational number, then the set of functions $f \in \mathfrak{G}^k$, such that Equation (3.2) has the Kupka-Smale property, is a generic subset of \mathfrak{G}^k .*

Proof : Equation (3.2) can be written in the frame of (2.4) with A defined by (2.2). Indeed, we set

$$B = \Delta^2, \quad D(B) = \{u \in \mathbb{H}^4(]0, 1[) / u(0) = u(1) = u_{xx}(0) = u_{xx}(1) = 0\} ,$$

and

$$\Gamma : \left(\begin{array}{ccc} \mathbb{H}^2(]0, 1[) \cap \mathbb{H}_0^1(]0, 1[) & \longrightarrow & \mathbb{H}^2(]0, 1[) \cap \mathbb{H}_0^1(]0, 1[) \\ v & \longmapsto & \gamma B^{-1}v + \kappa(v(d)) \end{array} \right) ,$$

where, for any $\beta \in \mathbb{R}$, $\kappa = \kappa(\beta)$ is the solution of

$$\left\{ \begin{array}{ll} \kappa_{xxxx}(x) = 0 & 0 < x < d \text{ and } d < x < 1 \\ \kappa(x) = \kappa_{xx}(x) = 0 & x = 0, 1 \\ \partial_x^k \kappa(d_-) = \partial_x^k \kappa(d_+) & k = 0, 1, 2 \\ \kappa_{xxx}(d_-) - \kappa_{xxx}(d_+) = -K\beta & \end{array} \right. .$$

For any $v \in \mathbb{H}^2(]0, 1[) \cap \mathbb{H}_0^1(]0, 1[)$, an easy computation gives

$$\langle \Gamma v | v \rangle_{D(B^{1/2})} = K|v(d)|^2 + \gamma \int_0^1 |v(x)|^2 dx .$$

Thus, the operators B and Γ clearly satisfy Hypotheses (B), (Gam) and (Loc). In [16], the constant γ was taken to be zero and the system (3.2) was not gradient. For this reason, we add a dissipative term γu_t , which gives a gradient structure to (3.2) and implies Property (UCP). Finally, by adapting the proofs of [16], we find that the rootvectors of A form a Riesz basis of X and that the eigenvalues of A satisfy

$$\lambda_{\pm n} = -\frac{\gamma}{2} \pm i(n\pi)^2 - K \sin^2(nd\pi) + O\left(\frac{1}{n}\right) .$$

If d belongs to \mathbb{Q} , the real parts of the eigenvalues have only a finite number of accumulation points, and so we have constructed an example satisfying a generalized condition (Spec). \square

3.3 Equations which satisfy only part of the hypotheses

Some examples of damped wave equations do not satisfy all the hypotheses of Theorem 2.6. Although we cannot prove the generic Kupka-Smale property, some of the propositions proved here are still valid. The hypotheses, which generally fail to be satisfied are (Grad), (UCP)b), (Spec)b) and (Spec)c). We point out that the assumptions (Spec)b) and (Spec)c) are crucial in the last steps of the proof of Theorem 2.6. However, such precise asymptotic estimates of the spectrum of A are not needed to prove Proposition 2.4. Actually, in the proof of Proposition 2.4, we only used the hypothesis (Spec)a) and the following property

(Spec') All the eigenvalues of A belong to a strip

$$\{z \in \mathbb{C} / -\beta < \operatorname{Re}(z) < -\alpha < 0\} ,$$

where α and β are two positive constants.

In this proof, the assumptions (Spec)b), (Spec)c) and (UCP)b) have been used only to show that (Spec') holds. Likewise, other interesting properties do not require these assumptions.

Theorem 3.2. *Let A be the operator defined by (2.2). Under the hypotheses (B), (Gam), (Spec)a), (Spec') and (UCP)a), the conclusions of Proposition 2.4, Proposition 2.5, Theorem 4.3 and Proposition 5.2 are still true.*

One application of this theorem is the beam equation with non-constant damping. In [29], Li, Yu, Liang and Zhu proved that the equation

$$\begin{cases} \left(\frac{\partial^2}{\partial t^2} + \gamma(x) \frac{\partial}{\partial t} + \frac{\partial^4}{\partial x^4} \right) u(x, t) = f(x, u), & (x, t) \in]0, 1[\times \mathbb{R}_+ \\ u(0, t) = \frac{\partial}{\partial x} u(0, t) = \frac{\partial^2}{\partial x^2} u(1, t) = \frac{\partial^3}{\partial x^3} u(1, t) = 0 \end{cases}$$

satisfies Hypotheses (Spec)a) and (Spec'), if an appropriate positivity condition on γ holds. They also enhanced that (Spec)b) is still not known in this case. As a consequence, Theorem 3.2 is valid for the beam equation with non-constant damping.

We end this section by remarking that there are other cases of equations, which do not exactly fit in our frame ; however, mimicing our proofs, we can show that the properties given in Theorem 3.2 are still true. This is for example the case for the beam equations with boundary damping, where one needs an additional equation on the boundary in order to describe the system (see [10]).

4 Proof of the main theorem : generic spectral properties

Let $(e, 0)$ be an equilibrium of Equation (2.4), that is a solution $e \in D(B)$ of the equation $Be = f(x, e)$. The linearized operator at the equilibrium $(e, 0)$ is

$$A_e \begin{pmatrix} u \\ v \end{pmatrix} = \left(A + \begin{pmatrix} 0 & 0 \\ f'_u(x, e) & 0 \end{pmatrix} \right) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -B(u + \Gamma v) + f'_u(x, e)u \end{pmatrix}, \quad (4.1)$$

with $D(A_e) = D(A)$. Notice that $(e, 0) \in X$ implies that e belongs to $\mathbb{H}^1(]0, 1[)$ and so $f'_u(x, e)$ is in $\mathcal{C}^0(]0, 1[)$.

In this section, we study generic properties of the spectrum of the linearized operator A_e . We need generic properties with respect to the function f , such as hyperbolicity of the equilibria $(e, 0)$ or simplicity of the eigenvalues of A_e . We point out that these

properties deserve a separate section as they do not only play an important role in the proof of Theorem 2.6, but have also their own interest. Their proofs, which mainly consist in applying Sard-Smale Theorem (Theorem A.2), need almost nothing else as the facts that the eigenfunctions of A_e have some smoothness and satisfy a unique continuation property such as (UCP)a). Thus, we would be able to prove the generic hyperbolicity of the equilibria and the generic simplicity of the eigenvalue for a larger class of operators. But, as this is not the main purpose of this paper, we keep most of the hypotheses of Theorem 2.6.

We only want to enhance that all the properties proved in this section are not specific to the one-dimensional case. For this reason, we do not assume Hypothesis (Spec), which seems to be an one-dimensional property. Instead, we will assume that A satisfies the following exponential decay property, which is often true in higher dimension.

(ED) There exist two positive constants δ_d and K_d such that, for all $U \in X$, and all $t > 0$,

$$\|e^{At}U\|_X \leq K_d e^{-\delta_d t} \|U\|_X .$$

Remarks :

- We assume Property (ED) to ensure the hyperbolicity of the equilibria. If Hypothesis (ED) does not hold, we can say that, generically with respect to f , the equilibria and the eigenvalues are simple. But, (ED) is needed to say that the equilibria are not only simple, but also hyperbolic.
- We will only use one property related to the one-dimensional case, which is the fact that $\mathbb{H}^1(]0, 1[)$ is continuously imbedded into $\mathcal{C}^0(]0, 1[)$. For the higher dimensional case, replacing our space $X = D(B^{1/2}) \times \mathbb{L}^2(]0, 1[)$ by an adequate subspace of $D(A^n)$ (n large enough), so that $D(A^n)$ is continuously imbedded in $\mathcal{C}^0(]0, 1[) \times \mathbb{L}^2(]0, 1[)$, we prove mutatis mutandis the same generic results. The only problem is that $(0, f(x, u))$ has to be in $D(A^n)$, and so we must be more careful and show that the perturbations of $(0, f(x, u))$ are still in $D(A^n)$. We refer to [6] where the cases of dimensions two and three are considered.

First, notice that an eigenvector of A_e corresponding to the eigenvalue λ is of the form $(\varphi, \lambda\varphi) \in D(A)$ and satisfies

$$-B(\varphi + \lambda\Gamma\varphi) + f'_u(x, e)\varphi = \lambda^2\varphi.$$

We will have to study functionals depending on φ and λ , so the last formulation is not very convenient as we must have $(\varphi + \lambda\Gamma\varphi) \in D(B)$, that is that φ belongs to a space which depends on the parameter λ . That is why we will use the equivalent formulation : $\varphi \in D(B^{1/2})$ and

$$\varphi + \lambda\Gamma\varphi = B^{-1} [(f'_u(x, e) - \lambda^2)\varphi] ,$$

which is much easier to handle.

Proposition 4.1. *Let A be the operator defined by (2.2), satisfying Properties (B), (Gam) and (UCP) defined in Section 2. If $(e, 0)$ is an equilibrium of (2.4), and $f \in \mathfrak{G}^k$, then the eigenvalues of A_e with non-negative real part are all real.*

Proof : As e is in $\mathbb{L}^\infty(]0, 1[)$, $f'_u(x, e(x))$ belongs to $\mathbb{L}^\infty(]0, 1[)$. Let $(\varphi, \lambda\varphi)$ be an eigenfunction of A_e that is

$$-B(\varphi + \lambda\Gamma\varphi) + f'_u(x, e)\varphi = \lambda^2\varphi.$$

By multiplying this equality by $\bar{\varphi}$ and integrating, we find

$$-\|\varphi\|_{D(B^{1/2})}^2 + \int_0^1 f'_u(x, e)|\varphi|^2 = \lambda^2\|\varphi\|_{\mathbb{L}^2}^2 + \lambda \langle \Gamma\varphi | \varphi \rangle_{D(B^{1/2})}.$$

If λ is not real, by taking the imaginary part of the above equality, we obtain

$$\operatorname{Re}(\lambda)\|\varphi\|_{\mathbb{L}^2}^2 = -\frac{1}{2} \langle \Gamma\varphi | \varphi \rangle_{D(B^{1/2})}.$$

We assumed in (UCP)b) that Γ is strictly positive on the eigenfunctions, which implies that if λ is not real, $\operatorname{Re}(\lambda) < 0$. \square

In what follows, we will denote f'_u for $f'_u(x, e)$ when no confusion is possible. Our last preliminary concerns the algebraic simplicity of the eigenvalues. Assume that there is an element $(u, v) \in D(A)$ such that

$$(A_e - \lambda Id) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \varphi \\ \lambda\varphi \end{pmatrix}, \text{ that is } \begin{cases} v - \lambda u = \varphi \\ -B(u + \Gamma v) + f'_u u - \lambda v = \lambda\varphi \end{cases}$$

So $v = \lambda u + \varphi$ and

$$u + \lambda\Gamma u + B^{-1}(\lambda^2 u - f'_u u) = -(\Gamma\varphi + 2\lambda B^{-1}\varphi).$$

We deduce that the algebraic multiplicity of λ is higher than one if and only if there exists an eigenvector $(\varphi, \lambda\varphi)$ with

$$\Gamma\varphi + 2\lambda B^{-1}\varphi \in R(Id + \lambda\Gamma + B^{-1}[(\lambda^2 - f'_u).]) \quad (4.2)$$

4.1 Genericity of the hyperbolicity

We recall that an equilibrium $(e, 0)$ of the wave equation (2.4) is hyperbolic if the spectrum of e^{A_e} does not intersect the unit circle of \mathbb{C} .

We recall also that \mathfrak{G}^k is the set of the functions $f \in \mathcal{C}^k([0, 1] \times \mathbb{R}, \mathbb{R})$ (with $k \geq 2$) endowed with the Whitney topology defined in Section 2.

Theorem 4.2. *Let A be the operator defined by (2.2), satisfying Properties (B), (Gam) and (UCP) defined in Section 2, and the exponential decay property (ED).*

Let \mathfrak{G}^H be the set of all functions $f \in \mathfrak{G}^k$, such that all the equilibria of the wave equation (2.4) are hyperbolic. Then \mathfrak{G}^H is a generic subset of \mathfrak{G}^k .

Proof : We assumed that A satisfies the exponential decay property (ED), which means that the radius of the spectrum of e^A is strictly less than one. Let U_0 be an element of X , and $U(t) = e^{At}U_0$. We have

$$e^{A_e}U = e^AU + \int_0^1 e^{A(1-t)}(0, f'_u(x, e)u(t))dt .$$

As $f'_u(x, e)u(s)$ is bounded in $\mathbb{H}^1(]0, 1[)$, which is compactly imbedded in $\mathbb{L}^2(]0, 1[)$, the operator e^{A_e} is a compact perturbation of e^A , and so the radius of its essential spectrum is the same as the one of e^A , thus it is strictly less than one.

Hence, concerning the hyperbolicity, we are reduced to consider the point spectrum. To show that no eigenvalue of e^{A_e} belongs to the unit circle, we have to prove that no eigenvalue of A_e belongs to the imaginary axis. By Proposition 4.1, the only possible eigenvalue of A_e on the imaginary axis is 0. In the case where 0 is an eigenvalue, there exists a function $\varphi \in D(B^{1/2})$ such that $\varphi = B^{-1}(f'_u\varphi)$. Notice that, as $Id - B^{-1}(f'_u\cdot)$ is a Fredholm operator of index 0, its injectivity is equivalent to its surjectivity.

The following proof can be found in [5]. We just give it in our frame for sake of completeness. Let \mathfrak{G}_n^H be the set of all functions $f \in \mathfrak{G}^k$ such that all equilibria $(e, 0)$ with $\|e\|_{\mathbb{L}^\infty} \leq n$ are hyperbolic. We only have to prove that \mathfrak{G}^H is a dense open subset of \mathfrak{G}^k , as $\mathfrak{G}^H = \bigcap_n \mathfrak{G}_n^H$. First, \mathfrak{G}_n^H is open. Indeed, if (f_k) is a sequence of functions of $\mathfrak{G}^k \setminus \mathfrak{G}_n^H$ which converges to some $f \in \mathfrak{G}^k$, then we have a sequence of equilibria $(e_k, 0)$ with $\|e_k\|_{\mathbb{L}^\infty} \leq n$, and a sequence of functions $\varphi_k \in D(B^{1/2})$ with $\|\varphi_k\|_{\mathbb{L}^2} = 1$, such that

$$e_k = B^{-1}f_k(x, e_k(x))$$

$$\varphi_k = B^{-1}((f_k)'_u(x, e_k(x))\varphi_k).$$

As $(e_k)_k$ is bounded in \mathbb{L}^∞ , the sequence $(f_k(x, e_k(x)))_k$ is bounded in \mathbb{L}^∞ too and thus (e_k) is bounded in $D(B^{1/2})$. Since $D(B^{1/2})$ is compactly imbedded in \mathbb{L}^∞ , by extracting a subsequence, we can assume that $e_k \rightarrow e$ in \mathbb{L}^∞ . Using the equation once more, we find that $e_k \rightarrow e$ in $D(B^{1/2})$. The same argument shows that $\varphi_k \rightarrow \varphi$ in $D(B^{1/2})$.

Continuity arguments imply that $(e, 0)$ is an equilibrium of Equation (2.4) corresponding to the nonlinearity f , that $\|e\|_{\mathbb{L}^\infty} \leq n$, and that $\varphi = B^{-1}((f'_u)(x, e(x))\varphi)$. This means that $f \notin \mathfrak{G}_n^H$.

We now have to prove the density of \mathfrak{G}_n^H . Let f be a function of \mathfrak{G}^k , let $\eta : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth compactly supported function which is equal to 1 on $[-n-1; n+1]$. It is sufficient to show that there exists a dense set of functions $b(x) \in \mathcal{C}^k$ such that $f(x, u) + b(x)\eta(u)$ is in \mathfrak{G}_n^H . We apply the Sard-Smale Theorem (see Theorem A.2) with

$$U = \{e \in D(B^{1/2}) / \|e\|_{\mathbb{L}^\infty} < n + 1\}, \quad V = \mathcal{C}^k([0, 1]), \quad Z = D(B^{1/2}),$$

and

$$\Phi(e, b)(x) = e - B^{-1}(f(x, e) + b(x)\eta(e)) = e - B^{-1}(f(x, e) + b(x)).$$

Let $z = 0$. If $(e, b) \in \Phi^{-1}(z)$, then $(e, 0)$ is an equilibrium, moreover, as we have noticed before, the surjectivity of $D_e\Phi(e, b) = Id - B^{-1}(f'_u(x, e))$ is equivalent to the hyperbolicity of $(e, 0)$. That is why, if the three hypotheses of Theorem A.2 are satisfied, \mathfrak{G}_n^H will be a generic, and a fortiori a dense subset of \mathfrak{G}^k .

We next verify that the hypotheses of Theorem A.2 hold. The space V is obviously separable and $D(B^{1/2})$ is separable since it is the image of the separable space $\mathbb{L}^2(]0, 1[)$ by the bounded operator $B^{-1/2}$. By Lemma 2.1, the operator $D_e\Phi = Id - B^{-1}(f'_u(x, e))$ is a Fredholm operator of index 0 from $D(B^{1/2})$ into itself. It remains to check that, if $(e, b) \in \Phi^{-1}(0)$, $D\Phi(e, b)$ is a surjective map from $D(B^{1/2}) \times V$ into $D(B^{1/2})$; that is, for any $g \in D(B^{1/2})$, we must find $(\varphi, c) \in D(B^{1/2}) \times V$ such that

$$D\Phi(e, b).(\varphi, c) = \varphi - B^{-1}(f'_u(\cdot, e)\varphi + c) = g.$$

According to Lemma 2.1, we need to find a function c such that for any ψ in the kernel of $Id - B^{-1}(f'_u(x, e))$, the function $g + B^{-1}c$ is orthogonal to ψ in $D(B^{1/2})$, that is that

$$\forall \psi \in \text{Ker}(Id - B^{-1}(f'_u(x, e))), \quad - \langle c | \psi \rangle_{\mathbb{L}^2} = \langle g | \psi \rangle_{D(B^{1/2})}.$$

As the kernel of $Id - B^{-1}(f'_u(x, e))$ is finite-dimensional, this is clearly possible. \square

4.2 Genericity of the simplicity of the eigenvalues

Let $(e, 0)$ be an equilibrium of (2.4). The simplicity of the eigenvalues of A_e is not directly required in the Kupka-Smale Property. But we will see that it plays a crucial role in the proof of Theorem 2.6 (see Theorem 4.4 or Proposition 5.3).

Theorem 4.3. *We assume that the operator A , defined by (2.2), satisfies Properties (B), (Gam) and (UCP) defined in Section 2, and the exponential decay property (ED).*

Let \mathfrak{G}^S be the set of all functions $f \in \mathfrak{G}^k$, such that, for any equilibrium $(e, 0)$ of (2.4), all the eigenvalues of the linearized operator A_e are simple. Then \mathfrak{G}^S is a generic subset of \mathfrak{G}^k .

Proof : Let $\mathfrak{G}_{n,m}^S$ be the set of all nonlinearities $f \in \mathfrak{G}^k$ such that for any equilibrium $(e, 0)$ of (2.4) with $\|e\|_{\mathbb{L}^\infty} \leq n$, all the eigenvalues λ with $|\lambda| \leq m$ of the linearized operator A_e are simple. As $\mathfrak{G}^S = \cap \mathfrak{G}_{n,m}^S$, we only need to prove that $\mathfrak{G}_{n,m}^S$ is an open dense set. We can prove that $\mathfrak{G}_{n,m}^S$ is an open set by proving that its complementary is closed by using the same method as in the proof of Theorem 4.2.

Like in the proof of Theorem 4.2, we apply Sard-Smale Theorem in order to prove the density. Let f be a function of \mathfrak{G}^k . By perturbing f , we can assume that f is of class \mathcal{C}^3 and belongs to \mathfrak{G}_{n+1}^H . Let η_1 and η_2 be two regular functions with compact support such that for any $u \in [-n-1, n+1]$, $\eta_1(u) = 1$ and $\eta_2(u) = u$. We apply Theorem A.2 in order to prove that we can find two functions b_1 and b_2 in $\mathcal{C}^k([0, 1])$ as small as wanted, such that $f(x, u) + b_1(x)\eta_1(u) + b_2(x)\eta_2(u)$ is in $\mathfrak{G}_{n,m}^S$. This is sufficient to prove that $\mathfrak{G}_{n,m}^S$ is a dense set. We set

$$\begin{aligned} U &= \{e \in D(B^{1/2}) / \|e\|_{\mathbb{L}^\infty} < n + 1\} \times (D(B^{1/2}) \setminus \{0\}) \times \mathbb{C}, \\ V &= \{b = (b_1, b_2) \in (\mathcal{C}^k([0, 1])^2 / f + b_1\eta_1 + b_2\eta_2 \in \mathfrak{G}_{n+1}^H\}, \\ Z &= (D(B^{1/2}))^2, \quad z = (0, 0). \end{aligned}$$

And we apply Theorem A.2 to the functional

$$\Phi(e, \varphi, \lambda, b)(x) = \begin{pmatrix} e - B^{-1}(f(x, e) + b_1(x) + b_2(x)e) \\ \varphi + \lambda\Gamma\varphi + B^{-1}((\lambda^2 - f'_u(x, e) - b_2(x))\varphi) \end{pmatrix}.$$

First, we notice that U and V are open subsets of separable metric spaces. Moreover, since f is of class \mathcal{C}^3 , Φ is of class \mathcal{C}^2 .

We next prove that $D\Phi$ is surjective from $(D(B^{1/2}))^2 \times \mathbb{C} \times (\mathcal{C}^k([0, 1])^2)$ into $(D(B^{1/2}))^2$ at each point of $\Phi^{-1}(0)$. More precisely, for each $(g, h) \in (D(B^{1/2}))^2$ and $(e, \varphi, \lambda, b) \in \Phi^{-1}(0)$, we must find $(\tilde{e}, \tilde{\varphi}, \tilde{\lambda}, \tilde{b}) \in (D(B^{1/2}))^2 \times \mathbb{C} \times (\mathcal{C}^k([0, 1])^2)$ such that $D\Phi.(\tilde{e}, \tilde{\varphi}, \tilde{\lambda}, \tilde{b}) = (g, h)$. We choose $\tilde{\lambda} = 0$ and $\tilde{b} = (-a(x)e(x), a(x))$ where $a \in \mathcal{C}^k([0, 1])$ has to be determined. Notice that $e = B^{-1}(f(x, e(x)))$, so, as $f \in \mathcal{C}^k$ and $B^{-1/2}$ is smoothing, e belongs to $\mathbb{H}^{k+1}([0, 1])$ and a fortiori to \mathcal{C}^k . So our choice $\tilde{b} = (-a(x)e(x), a(x))$ belongs as claimed to $(\mathcal{C}^k([0, 1])^2)$. We introduce the operator

$$L = Id + \lambda\Gamma + B^{-1}((\lambda^2 - f'_u - b_2).).$$

We have to find \tilde{e} , $\tilde{\varphi}$ and a such that

$$\tilde{e} - B^{-1}((f'_u(x, e) + b_2)\tilde{e}) = g$$

$$L\tilde{\varphi} = B^{-1}(f''_{uu}(x, e)\varphi\tilde{e}) + B^{-1}(a\varphi) + h.$$

Since the equilibrium $(e, 0)$ is hyperbolic, there exists $\tilde{e} \in D(B^{1/2})$ such that the first equality holds. By the Fredholm alternative given in Lemma 2.1, the second equality will

hold if we find a function $a \in \mathcal{C}^k([0, 1])$ such that $B^{-1}(f''_{uu}\varphi\tilde{e}) + B^{-1}(a\varphi) + h$ is orthogonal in $D(B^{1/2})$ to the kernel of \bar{L} , which is finite-dimensional. Let $(\varphi_1, \dots, \varphi_p)$ be a basis of this kernel. We have to find a function a such that

$$\forall i = 1, \dots, p, \quad \langle a\varphi | \varphi_i \rangle_{\mathbb{L}^2} = - \langle h + B^{-1}(f''_{uu}\varphi\tilde{e}) | \varphi_i \rangle_{D(B^{1/2})} := c_i. \quad (4.3)$$

We easily deduce from the unique continuation hypothesis (UCP)a) that the set $\{a\varphi / a \in \mathcal{C}^k([0, 1])\}$ is dense in $\mathbb{L}^2([0, 1])$, so we can find a function $a \in \mathcal{C}^k$ such that (4.3) is satisfied. So Hypothesis (ii) of Theorem A.2 is fulfilled.

It remains to prove that Hypothesis (i) of Theorem A.2 holds. We will show that, for any $(e, \varphi, \lambda, b) \in \Phi^{-1}(0)$, the operator

$$D_{(e, \varphi, \lambda)}\Phi : (\tilde{e}, \tilde{\varphi}, \tilde{\lambda}) \mapsto \begin{pmatrix} \tilde{e} - B^{-1}((f'_u(x, e) + b_2)\tilde{e}) \\ L\tilde{\varphi} + \tilde{\lambda}(\Gamma\varphi + 2\lambda B^{-1}\varphi) - B^{-1}(f''_{uu}\tilde{e}) \end{pmatrix}$$

is a Fredholm operator of index 1. Let m_λ be the multiplicity of the eigenvalue λ that is the dimension of the kernel of L . If $(\tilde{e}, \tilde{\varphi}, \tilde{\lambda})$ belongs to the kernel of $D_{(e, \varphi, \lambda)}\Phi$, then $\tilde{e} = 0$ since $(e, 0)$ is a hyperbolic equilibrium point. Hence we have

$$L\tilde{\varphi} = -\tilde{\lambda}(\Gamma\varphi + 2\lambda B^{-1}\varphi),$$

the dimension of the kernel of $D_{(e, \varphi, \lambda)}\Phi$ is m_λ if $(\Gamma\varphi + 2\lambda B^{-1}\varphi)$ does not belong to the range of L , and $m_\lambda + 1$ if it does. To determine the codimension of $D_{(e, \varphi, \lambda)}\Phi$, we use the same arguments once more. As $(e, 0)$ is a hyperbolic equilibrium, $(Id - B^{-1}((f'_u(x, e) + b_2)\cdot))$ is bijective ; hence, the codimension of the range of $D_{(e, \varphi, \lambda)}\Phi$ is equal to the codimension of the range of

$$(\tilde{\varphi}, \tilde{\lambda}) \mapsto L\tilde{\varphi} + \tilde{\lambda}(\Gamma\varphi + 2\lambda B^{-1}\varphi)$$

which is $m_\lambda - 1$ if $(\Gamma\varphi + 2\lambda B^{-1}\varphi)$ does not belong to the range of L , and m_λ if it does. In both cases, it follows that $D_{(e, \varphi, \lambda)}\Phi$ is a Fredholm operator of index 1. Thus, all the hypotheses of Theorem A.2 hold.

It follows, that for a generic set of functions $b = (b_1, b_2)$, for any (e, φ, λ) such that $(e, \varphi, \lambda, b) \in \Phi^{-1}(0)$, $D_{(e, \varphi, \lambda)}\Phi$ is surjective, that is, the codimension of its range is 0. This implies that $m_\lambda = 1$ and that $(\Gamma\varphi + 2\lambda B^{-1}\varphi)$ does not belong to the range of L . In other terms, this means that, for a generic set of functions (b_1, b_2) , for any equilibrium $(e, 0)$ of Equation (2.4), all the eigenvalues λ of A_e are geometrically and algebraically simple. \square

4.3 Genericity of the irrationality of some ratio

We will now prove that if $(e, 0)$ is a hyperbolic equilibrium of Equation (2.4), then the ratio between two distinct real eigenvalues of A_e is irrational for a generic set of nonlinearities

$f \in \mathfrak{G}^k$. Of course, this property is not really intuitive, but we use it in the proof of Theorem 2.6. Notice that, in [6], Brunovský and Raugel proved also that the ratio of a positive eigenvalue and the real part of one with negative real part is generically irrational. But, in our case, we could not prove it, that is why our generic result is a little weaker. However, this is not a problem since we could modify the method of [6].

We want also to point out that the proof of the following theorem is the only proof which presents complications due to the choice of dealing with an abstract frame. In the case of Equations (1.3) and (1.4), the proof is only slightly more involved than in the constant damping case. In order to prove the result in the abstract frame, we had to introduce the hypothesis (Loc), which is of course satisfied in the case of Equations (1.3) and (1.4). This hypothesis is used only in the proof of the following result.

We recall that C_{ev} is a real number defined in Hypothesis (Spec)b) of Section 2, but can obviously be replaced by any real number.

Theorem 4.4. *We assume that the operator A , defined by (2.2), satisfies Properties (B), (Gam), (UCP) and (Loc) defined in Section 2, and the exponential decay property (ED). Let \mathfrak{G}^I be the set of all functions $f \in \mathfrak{G}^k$ such that, for any equilibrium $(e, 0)$ of (2.4) and any two distinct real eigenvalues λ and μ of the linearisation A_e , there is no rational number $r \in \mathbb{Q}$ such that $\lambda = r\mu$ or $\lambda = rC_{ev}$. Then \mathfrak{G}^I is a generic subset of \mathfrak{G}^k .*

Proof : Let $\mathfrak{G}_{r,n,m}^I$ be the set of all the functions $f \in \mathfrak{G}^k$ such that, for any equilibrium $(e, 0)$ of (2.4) with $\|e\|_{\mathbb{L}^\infty} \leq n$, and any two distinct real eigenvalues λ and μ of the linearized operator A_e with $|\lambda| \leq m$ and $|\mu| \leq m$, we have $\frac{\lambda}{\mu} \neq r$ and $\frac{\lambda}{C_{ev}} \neq r$. As the set of the rational numbers r is countable, we only need to prove that $\mathfrak{G}_{r,n,m}^I$ is a dense open subset of \mathfrak{G}^k .

We can show that $\mathfrak{G}_{r,n,m}^I$ is open exactly as we do for \mathfrak{G}_n^H in the proof of Theorem 4.2. Here, to prove the density of $\mathfrak{G}_{r,n,m}^I$, we will not use the Sard-Smale theorem. We point out that the density can be proved by using a version of the Sard-Smale Theorem as it is done in [6]. But, we chose to use another method, which is possible once the generic simplicity of the eigenvalues is proved.

First notice that, as B and A have compact resolvents, there are only a finite number of equilibria $(e, 0)$ with $\|e\|_{\mathbb{L}^\infty} \leq n$ and a finite number of eigenvalues λ with $|\lambda| \leq m$. For this reason, we only have to prove that for any equilibrium $(e, 0)$ and any two real eigenvalues λ and μ , we can perturb f in such a way that the perturbed eigenvalues λ and μ satisfy $\lambda \neq r\mu$ and $\lambda \neq rC_{ev}$. First, by perturbing f , we can assume that $f \in \mathfrak{G}_{n+1,m+1}^S$, that is that λ and μ are simple. We consider perturbations of f of the form $f(x, u) + \tau a(x)(u(x) - e(x))$, where $a \in \mathcal{C}^k([0, 1])$ will be determined later. Notice that $(e, 0)$ is still an equilibrium of (2.4) for these perturbation, and that, as $\mathfrak{G}_{n+1,m+1}^S$ is open, there exists a number $\varepsilon > 0$, such that if $|\tau| < \varepsilon$, λ and μ are simple eigenvalues. By the implicit

function theorem, $\lambda(\tau)$ and $\mu(\tau)$ are \mathcal{C}^1 -functions of τ and the same property holds for the associated real normalized eigenvectors $(\varphi, \lambda\varphi)$ and $(\psi, \mu\psi)$ with $\|\varphi\|_{\mathbb{L}^2} = \|\psi\|_{\mathbb{L}^2} = 1$. We differentiate the equality

$$\varphi(\tau) + \lambda(\tau)\Gamma\varphi(\tau) + B^{-1}(\lambda^2(\tau) - f'_u(x, e) - \tau.a(x))\varphi(\tau) = 0$$

with respect to τ , to obtain that, at $\tau = 0$,

$$D_\tau\lambda(\Gamma\varphi + 2\lambda B^{-1}\varphi) = -(Id + \lambda\Gamma + B^{-1}(\lambda^2 - f'_u(x, e)))D_\tau\varphi + B^{-1}(a(x)\varphi). \quad (4.4)$$

The algebraic simplicity of λ implies that $(\Gamma\varphi(0) + 2\lambda B^{-1}\varphi(0))$ is not orthogonal to $\varphi(0)$ in $D(B^{1/2})$. By taking the scalar product in $D(B^{1/2})$ of the above equality with φ , we obtain :

$$D_\tau\lambda = \frac{\int_0^1 a(x)\varphi^2}{\langle \Gamma\varphi|\varphi \rangle_{D(B^{1/2})} + 2\lambda}.$$

Thus, we can easily find functions $a(x)$ (for example $a(x) = 1$) for which $D_\tau\lambda(0)$ is strictly positive. This means that, if f is such that $\lambda(0) = rC_{ev}$, by perturbing $f(x, u)$ by the function $\tau(u(x) - e(x))$, we have $\lambda(\tau) \neq rC_{ev}$ for τ small enough.

Assume now that $\lambda(0) = r\mu(0)$. We shall prove that there exists a perturbation of f of the form $\tau a(x)(u(x) - e(x))$ such that $\lambda(\tau) \neq r\mu(\tau)$ for τ small enough. We argue by contradiction : assume that, for any function $a \in \mathcal{C}^k([0, 1])$ and any $\tau \in]-\varepsilon; \varepsilon[$, we have $\lambda(\tau) = r\mu(\tau)$. This implies that, for any a and any τ , $D_\tau\lambda(\tau) = rD_\tau\mu(\tau)$, that is

$$\frac{\int_0^1 a(x)\varphi(\tau)^2}{\langle \Gamma\varphi(\tau)|\varphi(\tau) \rangle_{D(B^{1/2})} + 2\lambda(\tau)} = r \cdot \frac{\int_0^1 a(x)\psi(\tau)^2}{\langle \Gamma\psi(\tau)|\psi(\tau) \rangle_{D(B^{1/2})} + 2\mu(\tau)}.$$

This means that $\varphi(\tau)^2$ is proportionnal to $\psi(\tau)^2$, and because both are real normalized functions, we must have, for any $x \in [0, 1]$, $\varphi(\tau)^2(x) = \psi(\tau)^2(x)$. Thus

$$\langle \Gamma\psi(\tau)|\psi(\tau) \rangle_{D(B^{1/2})} + 2\mu(\tau) = r(\langle \Gamma\varphi(\tau)|\varphi(\tau) \rangle_{D(B^{1/2})} + 2\lambda(\tau)).$$

Since $r \neq 0$, by the hyperbolicity assumption, and since $r \neq 1$, as λ and μ are distinct eigenvalues, Assumption (Loc) of Theorem 2.6 implies that, for any function $a \in \mathcal{C}^k([0, 1])$ and any $\tau \in]-\varepsilon; \varepsilon[$,

$$2\lambda(\tau)(1 + \frac{1}{r}) + \langle \Gamma\varphi(\tau)|\varphi(\tau) \rangle_{D(B^{1/2})} = 0. \quad (4.5)$$

We will show that this is impossible. Indeed, we will prove that we can find a function $a \in \mathcal{C}^k([0, 1])$ such that the derivative of (4.5) satisfies

$$2D_\tau\lambda(0)(1 + \frac{1}{r}) + 2 \langle \Gamma\varphi(0)|D_\tau\varphi(0) \rangle_{D(B^{1/2})} \neq 0. \quad (4.6)$$

Thus, this will imply that we can perturb f in such a way that $\lambda(\tau) \neq r\mu(\tau)$ for τ small enough. To find a function a satisfying (4.6) we just have to work at the point $\tau = 0$, that is why, until the end of the proof, we omit the dependance in τ , and write for example λ instead of $\lambda(0)$.

First, we try to find a perturbation $a(x)$ such that $D_\tau\lambda = 0$ and $\langle \Gamma\varphi | D_\tau\varphi \rangle_{D(B^{1/2})} \neq 0$. Let $a(x)$ be in the orthogonal space of φ^2 in $\mathbb{L}^2(]0, 1[)$, which ensures that $D_\tau\lambda = 0$. As we have seen in the proof of Theorem 4.3, Hypothesis (UCP)a) implies that the set $\{a(x)\varphi / \langle a(x)\varphi | \varphi \rangle_{\mathbb{L}^2} = 0\}$ is dense in the orthogonal of φ in $\mathbb{L}^2(]0, 1[)$. In other words, the set $\{B^{-1}(a(x)\varphi) / \langle a(x)\varphi | \varphi \rangle_{\mathbb{L}^2} = 0\}$ is dense in the orthogonal of φ in $D(B^{1/2})$, which is exactly the range of $L = Id + \lambda\Gamma + B^{-1}(\lambda^2 - f'_u)$. Equality (4.4) together with the fact that $\|\varphi\|_{\mathbb{L}^2} = 1$ and $D_\tau\lambda = 0$, imply that $D_\tau\varphi$ is uniquely determined by

$$\begin{cases} L(D_\tau\varphi) = B^{-1}(a(x)\varphi) \\ \langle \varphi | D_\tau\varphi \rangle_{\mathbb{L}^2} = \langle B^{-1}\varphi | D_\tau\varphi \rangle_{D(B^{1/2})} = 0 . \end{cases}$$

We have seen that $\{B^{-1}(a(x)\varphi) / \langle a(x)\varphi | \varphi \rangle_{\mathbb{L}^2} = 0\}$ is dense in the range of L . So, we can find a perturbation $a(x)$ such that $\langle \Gamma\varphi | D_\tau\varphi \rangle_{D(B^{1/2})} \neq 0$ unless $\Gamma\varphi$ is proportionnal to $B^{-1}\varphi$. If this is the case, (4.5) implies that

$$\Gamma\varphi = -2\lambda(1 + \frac{1}{r})B^{-1}\varphi . \quad (4.7)$$

Finally, we will show that even if (4.7) holds, we can find a function $a(x) \in \mathcal{C}^k([0, 1])$ such that (4.6) is satisfied. Using (4.4) and (4.7), $D_\tau\varphi$ and $D_\tau\lambda$ are uniquely determined by

$$\begin{cases} L(D_\tau\varphi) - \frac{2\lambda}{r}D_\tau\lambda B^{-1}\varphi = B^{-1}(a(x)\varphi) \\ \langle \varphi | D_\tau\varphi \rangle_{\mathbb{L}^2} = 0 \end{cases} .$$

If we choose $a(x) = 1$, we obtain $D_\tau\lambda = -\frac{r}{2\lambda} \neq 0$ and $D_\tau\varphi = 0$, so Property (4.6) holds. \square

5 Proof of the main theorem : generic transversality

We begin this section with auxiliary results which will be used in the proof of Theorem 2.6. We would like to point out that these preliminary lemmas have some interest by themselves. Once they are obtained, we are able to apply the Brunovský-Poláčik-Raugel Theorem and thus to prove Theorem 2.6.

5.1 Preliminary lemmas

We recall that, if $(e, 0)$ is an equilibrium of (2.4), we set

$$A_e = \left(A + \begin{pmatrix} 0 & 0 \\ f'_u(x, e) & 0 \end{pmatrix} \right) .$$

The following proposition is a consequence of the fact that A_e is a compact perturbation of A .

Proposition 5.1. *If A satisfies Hypothesis (Spec) and if $(e, 0)$ is an equilibrium of (2.4) and $f \in \mathfrak{G}^k$, then A_e also satisfies Properties (Spec).*

Proof : Let (λ_n) be the eigenvalues of A and let $(V_{j,k}^n)$ be the associated rootvectors defined in Subsection 2.2. We assume that A satisfies Hypothesis (Spec). Let K be the function

$$K : \begin{pmatrix} X & \longrightarrow & X \\ (u, v) & \longmapsto & (0, f'_u(x, e(x))u) \end{pmatrix} .$$

As $D(B^{1/2})$ is compactly imbedded in $\mathbb{L}^2(]0, 1[)$, and $f'_u(x, e(x))$ belongs to $\mathbb{L}^\infty(]0, 1[)$, K is a compact operator. We want to prove that $A + K$ also satisfies Hypothesis (Spec). Such a property has been extensively studied for operators of the form $A + C$ where C is a small bounded perturbation. We will see that similar results hold for compact perturbations. Let $R(\lambda, A)$ (resp. $R(\lambda, A + K)$) be the resolvent of A (resp. $A + K$). As $(V_{j,k}^n)$ is a Riesz basis of X ,

$$V_{j,k}^n \rightarrow 0 \text{ weakly in } X \text{ when } n \rightarrow \infty .$$

Thus, because K is compact, we have

$$KV_{j,k}^n \rightarrow 0 \text{ when } n \rightarrow \infty . \quad (5.1)$$

For any $U \in X$, and any $\lambda \neq \lambda_n$, we introduce the sequences $(\alpha_{j,k}^n)$ and $(\beta_{j,k}^n)$ defined by

$$U = \sum_{n=0}^{+\infty} \sum_{j=0}^{m_n} \sum_{k=0}^{m_{n,j}} \alpha_{j,k}^n V_{j,k}^n ,$$

and

$$\begin{pmatrix} (\lambda_n - \lambda) & 1 & 0 & \dots & 0 \\ 0 & (\lambda_n - \lambda) & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & \dots & 0 & (\lambda_n - \lambda) \end{pmatrix} \begin{pmatrix} \beta_{j,0}^n \\ \beta_{j,1}^n \\ \vdots \\ \vdots \\ \beta_{j,m_{n,j}-1}^n \end{pmatrix} = \begin{pmatrix} \alpha_{j,0}^n \\ \alpha_{j,1}^n \\ \vdots \\ \vdots \\ \alpha_{j,m_{n,j}-1}^n \end{pmatrix} . \quad (5.2)$$

We have by construction

$$R(\lambda, A)U = \sum_n \sum_{j,k} \beta_{j,k}^n V_{j,k}^n .$$

Notice that, because of Hypothesis (Spec)b), the multiplicities $m_{n,j}$ are bounded. For $\lambda \neq \lambda_n$, Equality (5.2) implies that there exists a positive constant C , independent of j and n , such that

$$\sum_{k=0}^{m_{n,j}-1} |\beta_{j,k}^n|^2 \leq \frac{C}{|\lambda - \lambda_n|^2} \left(1 + \frac{1}{|\lambda - \lambda_n|^{2m_{n,j}-2}} \right) \sum |\alpha_{j,k}^n|^2 .$$

Using the equivalence of the norms (2.8), we find that there is a constant $C' > 0$ such that

$$\|KR(\lambda, A)U\|_X^2 \leq C' \max_{n,j,k} \left\{ \frac{\|KV_{j,k}^n\|_X^2}{|\lambda - \lambda_n|^2} \left(1 + \frac{1}{|\lambda - \lambda_n|^{2m_{n,j}-2}} \right) \right\} \|U\|_X^2 \quad (5.3)$$

Let $B(\lambda_n, r)$ be the ball of center λ_n and radius r in \mathbb{C} . Let α_{ev} be the constant introduced in Hypothesis (Spec). For $r < \alpha_{ev}$, $R(\lambda, A)$ is a well-defined bounded operator for any $\lambda \in \partial B(\lambda_n, r)$. Properties (5.1) and (5.3) imply that, for $r < \alpha_{ev}$,

$$\sup_{\lambda \in \partial B(\lambda_n, r)} \|KR(\lambda, A)\|_{\mathcal{L}(X)} \longrightarrow 0 , \quad \text{when } n \longrightarrow \infty .$$

That is why, for n large enough and for any $\lambda \in \partial B(\lambda_n, r)$, the operator $(\lambda + KR(\lambda, A))$ is invertible. Since we have $\lambda - (A + K) = (Id + KR(\lambda, A))(\lambda - A)$, the operator $(\lambda - (A + K))$ is invertible and

$$R(\lambda, A + K) = (\lambda - (A + K))^{-1} = R(\lambda, A)(Id + KR(\lambda, A))^{-1} .$$

Using the previous equality and arguing as in Theorems 3.16 and 3.18 of chapter IV of [25], we obtain that, for all $r_0 < \alpha_{ev}$, there exists a constant $C'' > 0$ such that, for n large enough, $r < r_0$ and $\lambda \in \partial B(\lambda_n, r)$, $R(\lambda, A + K)$ is a compact operator bounded by $\frac{C''}{r}$. Moreover, if (μ_n) is the sequence of eigenvalues of $A + K$, then

$$|\mu_n - \lambda_n| \longrightarrow 0 , \quad \text{as } n \longrightarrow \infty ,$$

and, since λ_n is simple for n large enough, so is μ_n . In conclusion, $A + K$ also satisfies Hypotheses (Spec)b) and (Spec)c).

To show that Hypothesis (Spec)a) also holds for $A + K$, we mimic the proofs concerning perturbations by small bounded operators. For example, adapting the proof of Theorem 4 of [30] (see also Th 4.15 of [25]), we show that the sequence of rootvectors of $A + K$ is equivalent to the one of A . More precisely, let $(U_i)_{i \geq 0}$, $(V_i)_{i \geq 0}$, $(U_i^*)_{i \geq 0}$ and $(V_i^*)_{i \geq 0}$, be respectively the rootvectors of A and $A + K$ and their biorthonormalized sequences (see

[14]). There exists an integer N such that, for $i \geq N$, the eigenvectors U_i and V_i correspond to simple eigenvalues. For $i \geq N$, let P_i and Q_i be the following eigenprojections

$$P_i = \langle \cdot | U_i^* \rangle U_i \quad Q_i = \langle \cdot | V_i^* \rangle V_i ,$$

and let P_0 (resp. Q_0) be the eigenprojection onto the space spanned by U_0, \dots, U_{N-1} (resp. V_0, \dots, V_{N-1}). The method given in the proof of Theorem 4 of [30] shows that there exists a bounded invertible operator D in $\mathcal{L}(X)$ such that, for $i = 0$ and $i \geq N$,

$$Q_i = D^{-1}P_iD .$$

Scaling in an appropriate way, we obtain for all i

$$V_i = D^{-1}U_i \text{ and } V_i^* = D^*U_i^* .$$

According to Theorem 2.1 of chapter VI of [14], these equalities imply that the sequence of rootvectors of $A + K$ forms a Riesz basis of X . \square

We recall that $S(t)$ denotes the local \mathcal{C}^0 -group generated by Equation (2.4). We denote by $S_u^*(t, s)$ the evolution operator defined by Equation (2.7).

Proposition 5.2. *Let f be a function of $\mathcal{C}^k([0, 1] \times \mathbb{R}, \mathbb{R})$ ($k \geq 1$). We assume that the hypotheses of Theorem 2.6 hold. If $\mathcal{I} \subset X$ is a bounded closed invariant set of $S(t)$ (for example a bounded complete solution), then, for any $U_0 \in \mathcal{I}$, the map $t \in \mathbb{R} \mapsto S(t)U_0 \in X$ is of class \mathcal{C}^k and $S(t)U_0$ is a classical solution. If $f(x, u)$ is analytic in $u \in \mathbb{R}$ uniformly in $x \in [0, 1]$, then the map $t \mapsto S(t)U_0$ is analytic. The same regularity properties also hold for $S_u^*(t, 0)$.*

Moreover, \mathcal{I} is bounded in $D(A)$ and $S(t)U_0$ belongs to $\mathcal{C}^{k-1}(\mathbb{R}, D(A))$. Furthermore, for any $f_0 \in \mathfrak{G}^k$ and any positive number R , there exists a neighborhood $N(f_0) \subset \mathfrak{G}^k$ and a positive constant $C = C(R, f_0)$, such that, if $f \in N(f_0)$ and \mathcal{I} is bounded in X by R , then for any $U_0 \in \mathcal{I}$,

$$\|S(t)U_0\|_{\mathcal{C}_b^0(\mathbb{R}, D(A)) \cap \mathcal{C}_b^1(\mathbb{R}, X)} \leq C(R, f_0).$$

Proof : This regularity property is a direct consequence of Theorem 1.1 of [20]. This theorem says that the trajectories in a compact invariant set are as regular as f (in the sense of our proposition) once we can find projections $(P_N)_{N \in \mathbb{N}}$, which commute with A and converge to the identity in X , such that AP_N is a bounded operator on X and the semigroup e^{At} is exponentially decreasing on $(Id - P_N)X$ for N large enough, with constants independent of N .

First notice that Proposition 2.5 implies that \mathcal{I} is compact.

Let $(V_{j,k}^n)$ be the Riesz basis composed of rootvectors of A . There exists a biorthonormalised basis $(W_{j,k}^n)_{n \in \mathbb{N}, j < m_n, k < m_{n,j}}$ such that

$$\langle V_{j,k}^n | W_{j',k'}^{n'} \rangle_X = \delta_{n,n'} \delta_{j,j'} \delta_{k,k'}$$

(see [14] for details). Let

$$P_N = \sum_{n=0}^N \sum_{j=0}^{m_n-1} \sum_{k=0}^{m_{n,j}-1} \langle \cdot | W_{j,k}^n \rangle V_{j,k}^n$$

be the projection onto the subspace generated by the eigenspaces corresponding to the first $N + 1$ eigenvalues. Obviously, P_N converges strongly to the identity, and AP_N is bounded. Since the exponential decay property has been proved in Proposition 2.4, all the hypotheses of Theorem 1.1 of [20] are satisfied once we have proved that $AP_N = P_N A$ in $D(A)$. Looking at

$$\langle A^* W_{j,k}^n | V_{j',k'}^{n'} \rangle = \langle W_{j,k}^n | A V_{j',k'}^{n'} \rangle,$$

we find that

$$A^* W_{j,k}^n = \begin{cases} \bar{\lambda}_n W_{j,k}^n & \text{if } k = m_{n,j} - 1 \\ \bar{\lambda}_n W_{j,k}^n + W_{j,k+1}^n & \text{if } 0 \leq k \leq m_{n,j} - 2. \end{cases} \quad (5.4)$$

Then, an easy computation gives the expected commutation between A and P_N .

We prove with the same arguments that the above properties are satisfied by $S_u^*(t, 0)$.

Finally, notice that the second part of our proposition is also a consequence of Theorem 1.1 of [20]. Although these properties are not enhanced in the statement of Theorem 1.1, they can be deduced from its proof. \square

Let M be a positive constant. Let $f \in \mathfrak{G}^k$ and let (u, u_t) be a complete bounded solution of (2.4) with $\|u(t)\|_{\mathbb{L}^\infty} \leq M$ for all time $t \in \mathbb{R}$. As the system is gradient and asymptotically smooth, there are two distinct equilibria e_+ and e_- such that $(u, u_t) \rightarrow (e_\pm, 0)$ when $t \rightarrow \pm\infty$.

Proposition 5.3. *We assume that the hypotheses of Theorem 2.6 are fulfilled.*

Let f be a function in \mathfrak{G}^k such that $f(x, u)$ is analytic with respect to $u \in [-M, M]$, uniformly in x , and such that, for any equilibrium $(e, 0)$ of (2.4), $(e, 0)$ is hyperbolic and all the eigenvalues of the linearized operator A_e are simple. Let (u, u_t) be the trajectory defined above. Then, there exist a positive eigenvalue λ of A_{e_-} with corresponding eigenfunction $(\varphi, \lambda\varphi)$, a nonzero number b and a positive constant δ such that

$$\left\| \begin{pmatrix} u \\ u_t \end{pmatrix} - \begin{pmatrix} e_- \\ 0 \end{pmatrix} - b e^{\lambda t} \begin{pmatrix} \varphi \\ \lambda\varphi \end{pmatrix} \right\|_X = o(e^{(\lambda+\delta)t}) \quad \text{when } t \rightarrow -\infty.$$

Moreover,

$$\|u_t - b\lambda e^{\lambda t} \varphi\|_{D(B^{1/2})} = o(e^{(\lambda+\delta)t}) \quad \text{when } t \rightarrow -\infty.$$

Proof : The first property is classical since the spectrum of A_{e_-} contains only a finite number of eigenvalues with positive real part. To obtain the second estimate, we use the regularity proposition 5.2 to differentiate Equation (2.4) with respect to the time variable. Then the second estimate is shown as the first one. \square

The next proposition gives a similar result for the adjoint equation (2.7). However, new difficulties come from the fact that the concerned part of the spectrum contains an infinite number of eigenvalues.

Let f be as in Proposition 5.3 (actually, $f \in \mathfrak{G}^k$ with $k \geq 3$ is enough). Let (u, u_t) be a complete bounded solution of Equation (2.4), that is a solution of (2.4) which is uniformly bounded in X for all time $t \in \mathbb{R}$. Let (θ, ψ) be a complete bounded solution of the adjoint equation (2.7), that is

$$\begin{cases} \theta_t = \psi - B^{-1}(f'_u(x, u)\psi) \\ \psi_t = -B(\theta - \Gamma\psi) \end{cases} . \quad (5.5)$$

Proposition 5.2 implies that (θ, ψ) belongs to $\mathcal{C}^2(\mathbb{R}, X) \cap \mathcal{C}^1(\mathbb{R}, D(A^*))$. We deduce from (5.5) that

$$\psi_{tt} = -B(\psi - B^{-1}(f'_u(x, u)\psi) - \Gamma\psi_t) .$$

This can be written

$$\frac{\partial}{\partial t} \begin{pmatrix} \psi \\ \psi_t \end{pmatrix} = - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left[A + \begin{pmatrix} 0 & 0 \\ f'_u(x, u) & 0 \end{pmatrix} \right] \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \psi \\ \psi_t \end{pmatrix} . \quad (5.6)$$

Proposition 5.2 implies that (ψ, ψ_t) belongs to $\mathcal{C}^1(\mathbb{R}, X) \cap \mathcal{C}^0(\mathbb{R}, D(A^*))$. Thus, we can formulate the following proposition.

Proposition 5.4. *Let f and (u, u_t) be as in Proposition 5.3. If (θ, ψ) is a complete bounded solution of the adjoint equation (2.7), there exists a positive real number μ such that*

$$\lim_{t \rightarrow -\infty} \ln \left\| \begin{pmatrix} \psi \\ \psi_t \end{pmatrix} \right\|_X^{1/t} = \mu .$$

Moreover, there exist a positive constant δ and a solution

$$(\psi^\infty, \psi_t^\infty) \in \mathcal{C}^1(\mathbb{R}, X) \cap \mathcal{C}^0(\mathbb{R}, D(A^*))$$

of the limit equation

$$\frac{\partial}{\partial t} \begin{pmatrix} \psi^\infty \\ \psi_t^\infty \end{pmatrix} = - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A_{e_-} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \psi^\infty \\ \psi_t^\infty \end{pmatrix} ,$$

such that

$$\left\| \begin{pmatrix} \psi \\ \psi_t \end{pmatrix} - \begin{pmatrix} \psi^\infty \\ \psi_t^\infty \end{pmatrix} \right\|_X = o(e^{(\mu+\delta)t}) \quad \text{when } t \longrightarrow -\infty.$$

In particular

$$\lim_{t \rightarrow -\infty} \ln \left\| \begin{pmatrix} \psi^\infty \\ \psi_t^\infty \end{pmatrix} \right\|_X^{1/t} = \mu .$$

Proof : We refer here to the proof of Propositions 5.3 and 5.4 of [6], which are proved by using Theorems B.5, B.6 and B.7 of [6]. The theorems of Appendix B of [6] give sufficient conditions, under which a solution of an equation of type $V_t = C(t)V$, with $C(t) \longrightarrow C(\infty)$ when $t \longrightarrow \infty$, converges to a solution of the limit equation $\Psi_t = C(\infty)\Psi$.

We only want to enhance that Hypothesis (Spec) implies that for any real number l , we can find a gap in the real part of the spectrum as near as needed from l . This property is necessary to define the spectral projections and to apply the theorems of Appendix B of [6]. \square

The last proposition will be used to find a point $x \in [0, 1]$ such that the asymptotic speed of a trajectory (ψ, ψ_t) of Equation (5.6) in X , and the asymptotic speed of the function $\psi(x, \cdot)$ at the chosen point x are equal.

Proposition 5.5. *Let $(c_n)_{n \in \mathbb{N}}$ be a sequence of nonzero complex numbers in $\ell^1(\mathbb{C})$, and let $(\lambda_n)_{n \in \mathbb{N}}$ be a sequence of complex numbers such that $\sup\{\operatorname{Re}(\lambda_n)\} < \infty$. Let*

$$f(t) = \sum_{n \in \mathbb{N}} c_n e^{\lambda_n t} .$$

The function f is well defined since $(c_n)_{n \in \mathbb{N}} \in \ell^1(\mathbb{C})$ and since for each t , $|e^{\lambda_n t}|$ is uniformly bounded with respect to n . Moreover, we have

$$\inf \left\{ \lambda \in \mathbb{R} / \lim_{t \rightarrow \infty} |f(t)| e^{-\lambda t} = 0 \right\} = \sup\{\operatorname{Re}(\lambda_n)\}.$$

Proof : It is clear that if $\lambda > \sup\{\operatorname{Re}(\lambda_n)\}$, then $|f(t)| e^{-\lambda t} \longrightarrow 0$. So

$$\inf \left\{ \lambda \in \mathbb{R} / \lim_{t \rightarrow \infty} |f(t)| e^{-\lambda t} = 0 \right\} \leq \sup\{\operatorname{Re}(\lambda_n)\}.$$

Now, assume that there exist a number λ and a positive constant ε such that $\lambda < \lambda + \varepsilon < \sup\{\operatorname{Re}(\lambda_n)\}$ and $|f(t)| e^{-\lambda t} \longrightarrow 0$. As $f(t) e^{-(\lambda+\varepsilon)t} = o(e^{-\varepsilon t})$, the Laplace transform of f ,

$$Lf(z) = \int_0^\infty f(t) e^{-zt} dt ,$$

is defined on the half-plane $H = \{z \in \mathbb{C} / \operatorname{Re}(z) \geq \lambda + \varepsilon\}$ and is holomorphic on H . But if z is such that $\operatorname{Re}(z) > \sup\{\operatorname{Re}(\lambda_n)\}$, then we can develop Lf as a sum of meromorphic functions as follows :

$$Lf(z) = \int_0^\infty \sum c_n e^{(\lambda_n - z)t} = \sum c_n \int_0^\infty e^{(\lambda_n - z)t} = - \sum \frac{c_n}{\lambda_n - z}.$$

As Lf is holomorphic on H , this expression must be valid on the whole half-plane H . But, because $\sup\{\operatorname{Re}(\lambda_n)\} > \lambda + \varepsilon$ and $c_n \neq 0$, Lf has poles in H , which contradicts the fact that Lf is holomorphic in H . So,

$$\inf \left\{ \lambda \in \mathbb{R} / \lim_{t \rightarrow \infty} |f(t)| e^{-\lambda t} = 0 \right\} \geq \sup\{\operatorname{Re}(\lambda_n)\}.$$

□

5.2 Proof of Theorem 2.6

In this subsection, we prove Theorem 2.6 by using the Brunovský-Poláčik-Raugel Theorem, recalled in Appendix A. The way to apply it has already been explained in [5] and [6]. Whereas the verification of the hypotheses (h1)-(h6) does not change, significant changes appear in verifying condition (h7).

First step : Application of Theorem A.1

We have to put our problem in the framework of Theorem A.1. Let $Z = \mathbb{L}^\infty(]0, 1[) \times \mathbb{L}^2(]0, 1[)$, let $\Lambda = \mathfrak{G}^k$ ($k \geq 2$), $U = (u, u_t)$ and $F((u, v), f) = (0, f(x, u))$. Equation (2.4) becomes

$$U_t = AU + F(U, f).$$

We set for any $r \in \mathbb{N}$,

$$\Lambda^r = \mathcal{C}^r([0, 1] \times [-n, n]).$$

Let R be the restriction operator

$$Rf = f|_{[0,1] \times [-n,n]},$$

which is continuous, open and surjective. We recall that, in the proof of Theorem 4.2, we have introduced the set \mathfrak{G}_n^H of all the functions f such that all the equilibria $(e, 0)$ of (2.4) with $\|e\|_{\mathbb{L}^\infty} \leq n$ are hyperbolic. We proved that \mathfrak{G}_n^H is an open dense subset of \mathfrak{G}^k . Let L be the open dense subspace of Λ^k defined by

$$L = R\mathfrak{G}_n^H.$$

We also set $M = n$. Let \mathfrak{G}_n^{KS} be the set of all the functions in \mathfrak{G}^k for which all the heteroclinic orbits (u, u_t) of Equation (2.4) with $\|(u, u_t)\|_Z < n$ are transverse. Assume that Theorem A.1 can be applied. If \mathcal{L} is the generic subset of L given in the conclusion of Theorem A.1, then $R^{-1}\mathcal{L} \subset \mathfrak{G}_n^{KS}$ is a generic subset of \mathfrak{G}_n^H and so a generic subset of \mathfrak{G}^k . As

$$\mathfrak{G}^{KS} = \bigcap_{n \in \mathbb{N}} \mathfrak{G}_n^{KS},$$

Theorem 2.6 will be proved. Thus, it remains to prove that all the hypotheses of Theorem A.1 are satisfied.

Hypotheses (FP), (AP), (BUP1), (BUP2), (h1), (h2), (h3), (h4) of Theorem A.1 are obvious or were proved in Subsections 2.1 and 2.2. Condition (h6) is a consequence of Proposition 5.2 ; (h5) comes from the gradient structure and the asymptotic smoothness of (2.4). Finally, we have only to prove that Hypothesis (h7) is satisfied.

Let $\lambda_0 = f_0$ and let \mathcal{V} be a neighborhood of f_0 in L . Let $(e, 0)$ be an equilibrium of (2.4) with $\|e\|_{\mathbb{L}^\infty} < M$. We recall that the Morse index of $(e, 0)$ is the dimension of its unstable manifold, that is the number of eigenvalues with positive real part of A_e . As f_0 belongs to $\mathcal{C}^0([0, 1] \times [-n, n])$ and $(e, 0)$ satisfies the equality $e = B^{-1}f_0(x, e)$, the equilibrium $(e, 0)$ is bounded in $D(A)$ by a constant which only depends on M and f_0 . As the equilibria of (2.4) have the hyperbolicity property, they are isolated in X . So, due to the compact imbedding of $D(A)$ into X , the number of equilibria $(e, 0)$ of (2.4) with $\|e\|_{\mathbb{L}^\infty} < M$ is finite. We deduce that there exists an integer r such that $r - 1$ is strictly larger than all the Morse indices of the equilibria of (2.4). Then, we set

$$\hat{\Lambda} = \Lambda^r.$$

The Morse indices depend continuously on the non-linearity f . As the number of equilibria is finite, we can restrict the neighborhood \mathcal{V} without loss of generality, such that for any $f \in \mathcal{V}$, the Morse indices of the equilibria stay strictly less than $r - 1$. By density, we can find a function $f_1 \in \mathcal{V}$ which belongs to $R\mathfrak{G}_n^H \cap \Lambda^r$. As $R\mathfrak{G}_n^H \cap \Lambda^r \cap \mathcal{V}$ is open in Λ^r , by a simple perturbation, we can find a function $f_2 \in R\mathfrak{G}_n^H \cap \Lambda^r \cap \mathcal{V}$ which is analytic in u , uniformly with respect to x . In the proof of Theorems 4.3 and 4.4, we showed that for any $f \in \mathfrak{G}^k$, we can find a perturbation of the form $\eta(a(x) + b(x)u)$, with a and b as smooth as f , such that $f(x, u) + a(x) + b(x)u$ belongs to $\mathfrak{G}^I \cap \mathfrak{G}^S$. That is why, by perturbing f_2 in this way, we can find $\hat{f} \in R\mathfrak{G}_n^H \cap \Lambda^r \cap \mathcal{V}$, which belongs to $\mathfrak{G}^I \cap \mathfrak{G}^S$ and which is analytic in u , uniformly with respect to x . Indeed, f_2 was assumed to be analytic in u , uniformly with respect to x , and a perturbation of the form $\eta(a(x) + b(x)u)$ does not alter this property. By construction, Hypotheses (h7)(a) and (h7)(b) of Theorem A.1 hold. It remains to prove that Hypothesis (h7)(c) is satisfied for our \hat{f} .

Second step : Hypothesis (h7)(c)

Let (u, u_t) be a heteroclinic solution of Equation (2.4) with $\sup_{t \in \mathbb{R}} \|(u, u_t)\|_Z < n$ and (θ, ψ) be a nontrivial complete bounded solution of the adjoint equation (2.7). We must find a function $h \in \mathcal{C}^r([0, 1] \times \mathbb{R}, \mathbb{R})$ such that

$$I = \int_0^\infty \left\langle (\theta, \psi) \left| D_f F(U, \hat{f}) h \right. \right\rangle_X dt \neq 0,$$

that is such that

$$I = \int_0^\infty \int_0^1 \psi(x, t) h(x, u(x, t)) dx dt \neq 0.$$

In the particular case where $h(x, u) = b(x)g(u)$, the above condition becomes

$$I = \int_0^1 b(x) \left(\int_{\mathbb{R}} \psi(x, t) g(u(x, t)) dt \right) dx \neq 0. \quad (5.7)$$

So it is sufficient to find a point x and a function $g \in \mathcal{C}^r(\mathbb{R}, \mathbb{R})$ such that

$$J = \int_{\mathbb{R}} \psi(x, t) g(u(x, t)) dt \neq 0. \quad (5.8)$$

As u is a heteroclinic solution, there are two distinct equilibria e_+ and e_- such that $u \rightarrow e_\pm$ when $t \rightarrow \pm\infty$. We choose x such that $(e_+ - e_-)(x) \neq 0$ and all the eigenfunctions $(\varphi, \lambda\varphi)$ of A_{e_-} satisfy $\varphi(x) \neq 0$. This is possible due to Hypothesis (UCP)a) of Theorem 2.6 and the fact that the set of eigenfunctions is a countable set. Such a choice is essential in the remaining part of the proof. Indeed, we will see that it ensures that the asymptotic behaviour in time of the real functions $u(x, t)$ and $\psi(x, t)$ is similar to those of $u(\cdot, t)$ and $\psi(\cdot, t)$ in $D(B^{1/2})$. To the end of the proof, we will only consider the functions at this chosen point x .

As $\hat{f}(\xi, u)$ is analytic in u , uniformly in ξ , we can apply Proposition 5.2, and so $u(x, \cdot)$ and $\psi(x, \cdot)$ are analytic functions of t . Due to Proposition 5.3, there exist a nonzero number b and a positive real eigenvalue λ of A_{e_-} with eigenvector $(\varphi, \lambda\varphi)$ such that

$$\left\| \begin{pmatrix} u \\ u_t \end{pmatrix} - \begin{pmatrix} e_- \\ 0 \end{pmatrix} - be^{\lambda t} \begin{pmatrix} \varphi \\ \lambda\varphi \end{pmatrix} \right\|_X = o(e^{(\lambda+\delta)t}).$$

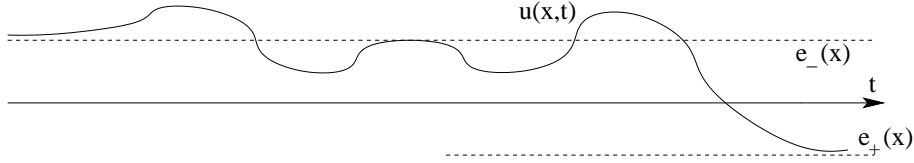
As $D(B^{1/2})$ is imbedded in \mathcal{C}^0 , we obtain

$$u(x, t) = e_-(x) + be^{\lambda t}\varphi(x) + o(e^{(\lambda+\delta)t}).$$

Using the second estimate of Proposition 5.3, we also have

$$u_t(x, t) = b\lambda\varphi(x)e^{\lambda t} + o(e^{(\lambda+\delta)t}),$$

when t goes to $-\infty$. Because of the choice of x , $\varphi(x) \neq 0$, so we know that $u_t(x, t)$ does not vanish on a neighborhood of $-\infty$. Without loss of generality, we can assume for example that $b\varphi(x) > 0$ and $e_-(x) > e_+(x)$. To summarize, $u(x, t)$ looks like



We choose the function g of the form

$$g(u) = g_{\zeta, \varepsilon}(u) = \frac{1}{\varepsilon} \Theta\left(\frac{u - \zeta}{\varepsilon}\right),$$

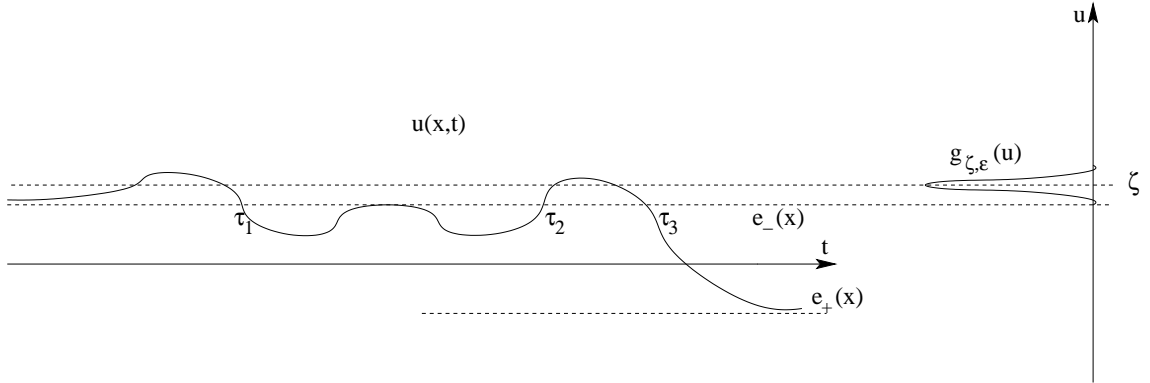
where Θ is a smooth normalized bump function. For example, we take

$$\Theta(s) = \begin{cases} 0 & \text{if } |s| > 1 \\ \frac{1}{C} e^{-\frac{1}{1-s^2}} & \text{if } |s| \leq 1 \end{cases},$$

where

$$C = \int_{-1}^1 e^{-\frac{1}{1-s^2}} d\sigma.$$

In what follows, we always assume that $0 < \varepsilon < \zeta - e_-(x)$. As $u(x, \cdot)$ is strictly increasing in a neighborhood of $t = -\infty$, $e_-(x) \neq e_+(x)$, and that $u(x, \cdot)$ is analytic in time, there is only a finite number of solutions of the equation $u(x, \tau) = e_-(x)$. We denote by τ_1, \dots, τ_m all the solutions τ of $u(x, \tau) = e_-(x)$, for which we do not have $u(x, t) \leq e_-(x)$ in a neighborhood of $t = \tau$.



Let $(N_i)_{i=0, \dots, m}$ be disjoint neighborhoods of $t = -\infty$ for $i = 0$ and τ_i for $i = 1, \dots, m$; then for $\zeta - e_-(x)$ and ε small enough we can split the integral J into a finite sum of integrals :

$$J = \int_{\mathbb{R}} \psi(x, t) g(u(x, t)) dt = \sum_{i=0}^m \int_{N_i} \psi(x, t) g_{\zeta, \varepsilon}(u(x, t)) dt = \sum_{i=0}^m J_i.$$

In their paper [5], Brunovský and Poláčik conclude quickly from this splitting, as a property of the parabolic equation ensures that $\frac{d}{dt}u(x, \tau_i) \neq 0$ for a generic x . In our case, we cannot be sure that $\frac{d}{dt}u(x, \tau_i) \neq 0$, so we need to estimate the integrals J_i in order to conclude. This method was introduced by Brunovský and Raugel in [6].

Third step : Estimations of the integrals J_i

Lemma 5.6. *If $0 < \varepsilon < \zeta - e_-(x)$ with $\zeta - e_-(x)$ small enough, there exists a rational number $r \in \mathbb{Q}$ such that*

$$\sum_{i=1}^m J_i = S((\zeta - e_-(x))^r) + \omega(\zeta, \varepsilon),$$

where $S(z)$ is a power series of z and

$$\lim_{\varepsilon \rightarrow 0} \omega(\zeta, \varepsilon) = 0.$$

Proof : For sake of completeness, we repeat here the proof of Lemma 5.6 of [6].

As $u(x, \cdot)$ and $\psi(x, \cdot)$ are analytic functions of the time, we may write, when t is near τ_i ,

$$u(x, t) = e_-(x) + \sum_{l=k}^{+\infty} a_l (t - \tau_i)^l,$$

and

$$\psi(x, t) = \sum_{l=k'}^{+\infty} d_l (t - \tau_i)^l.$$

In what follows, we assume that k is odd. Denote

$$z = (\zeta - e_-(x))^{-1/k} (t - \tau_i), \tag{5.9}$$

then, for $\zeta \neq e_-(x)$, $u(t) = \zeta$ if and only if

$$a_k (\zeta - e_-(x)) z^k + \sum_{l=k+1}^{\infty} a_l (\zeta - e_-(x))^{\frac{l}{k}} z^l = \zeta - e_-(x),$$

or,

$$H(z, (\zeta - e_-(x))^{1/k}) \equiv a_k z^k + \sum_{l=k+1}^{\infty} a_l (\zeta - e_-(x))^{\frac{l}{k}-1} z^l = 1.$$

Since $a_k \neq 0$, we may apply the implicit function theorem to the equation

$$H(z, (\zeta - e_-(x))^{1/k}) - 1 = 0$$

in the neighbourhood of $(a_k^{-1/k}, 0)$. Hence, locally near $z = a_k^{-1/k}$, the above equation has a unique solution

$$z = a_k^{-1/k} + \sum_{l=1}^{+\infty} c_l (\zeta - e_-(x))^{l/k} .$$

Let t_ζ be the solution of $u(x, t_\zeta) = \zeta$ near τ_i . Substituting the above expression of z into (5.9), we obtain that, for ζ close to $e_-(x)$,

$$t_\zeta = \tau_i + a_k^{-1/k} (\zeta - e_-(x))^{1/k} + (\zeta - e_-(x))^{1/k} S((\zeta - e_-(x))^{1/k}) ,$$

where $S(z)$ is a power series of z . Further, we deduce that

$$\begin{aligned} u_t(t_\zeta) &= k a_k (t_\zeta - \tau_i)^{k-1} + \sum_{l=k+1}^{+\infty} l a_l (t_\zeta - \tau_i)^{l-1} \\ &= k a_k^{1-1/k} (\zeta - e_-(x))^{1-1/k} S((\zeta - e_-(x))^{1/k}) . \end{aligned}$$

Due to the change of variables $t = t(u)$ in J_i and the fact that $g_{\zeta, \varepsilon}$ converges to the Dirac function at the point ζ when $\varepsilon \rightarrow 0$, we obtain

$$\begin{aligned} J_i &= \int_{|u(t)-\zeta| \leq \varepsilon} \psi(x, t) g(u(t)) dt \\ &= \int_{\zeta-\varepsilon}^{\zeta+\varepsilon} \psi(x, t(u)) g_{\zeta, \varepsilon}(u) \frac{du}{u_t(x, t(u))} \\ &= \frac{\psi(x, t_\zeta)}{u_t(x, t_\zeta)} + \omega(\zeta, \varepsilon) , \end{aligned}$$

with $\omega(\zeta, \varepsilon) \rightarrow 0$ when $\varepsilon \rightarrow 0$. Finally, using the analyticity of $u(x, \cdot)$ and $\psi(x, \cdot)$ we obtain that

$$J_i = (\zeta - e_-(x))^{\frac{k'+1}{k}-1} S((\zeta - e_-(x))^{\frac{1}{k}}) + \omega(\zeta, \varepsilon).$$

In the case where k is even, the only difference would be that we must split J_i into two parts, the one when $t < \tau_i$ and the one when $t > \tau_i$. Next, we can deal with each part as we do with the whole integral J_i when k is odd (see [6]). \square

We recall that when t goes to $-\infty$ and $\delta > 0$ is small enough

$$u(x, t) = e_-(x) + b e^{\lambda t} \varphi(x) + o(e^{(\lambda+\delta)t}),$$

and

$$u_t(x, t) = b \lambda \varphi(x) e^{\lambda t} + o(e^{(\lambda+\delta)t}).$$

Moreover, according to Proposition 5.4, there exist a positive real number μ and a solution $(\psi^\infty, \psi_t^\infty) \in \mathcal{C}^0(\mathbb{R}, D(A^*))$ of the limit equation

$$\frac{\partial}{\partial t} \begin{pmatrix} \psi^\infty \\ \psi_t^\infty \end{pmatrix} = - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A_{e_-} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \psi^\infty \\ \psi_t^\infty \end{pmatrix}$$

such that

$$\lim_{t \rightarrow -\infty} \ln \left\| \begin{pmatrix} \psi^\infty \\ \psi_t^\infty \end{pmatrix} \right\|_X^{1/t} = \mu,$$

and

$$\psi(x, t) = \psi^\infty(x, t) + O(e^{(\mu+\delta)t}),$$

when t goes to $-\infty$ and $\delta > 0$ is small enough.

Lemma 5.7. *If $0 < \varepsilon < \zeta - e_-(x)$ are small enough, then*

$$\begin{aligned} J_0 &= \frac{\psi^\infty(t_\zeta) + O\left((\zeta - e_-(x))^{\frac{\mu+\delta}{\lambda}}\right)}{u_t(t_\zeta)} + \omega(\zeta, \varepsilon) \\ &= \frac{\psi^\infty(t_\zeta)}{\lambda\phi(x)(\zeta - e_-(x))} + O\left(\frac{\psi^\infty(t_\zeta)}{(\zeta - e_-(x))^{1-\frac{\delta}{\lambda}}}\right) + O\left((\zeta - e_-(x))^{\frac{\mu}{\lambda}-1+\frac{\delta}{\lambda}}\right) + \omega(\zeta, \varepsilon) \end{aligned}$$

where

$$t_\zeta = \frac{1}{\lambda} \ln \left[\frac{\zeta - e_-(x)}{b\phi(x)} + O(\zeta - e_-(x))^2 \right], \quad (5.10)$$

and

$$\lim_{\varepsilon \rightarrow 0} \omega(\zeta, \varepsilon) = 0.$$

Proof : The proof is exactly the same as the one of Lemma 5.6. Here, the implicit function theorem gives us that, if t_ζ is the unique solution in N_0 of $u(x, t) = \zeta$, then

$$t_\zeta = \frac{1}{\lambda} \ln \left[\frac{\zeta - e_-(x)}{b\phi(x)} + O(\zeta - e_-(x))^2 \right].$$

Finally, we use the same change of variables as in Lemma 5.6 to obtain the result (see Lemma 5.7 of [6]). \square

We recall that, by the choice of \hat{f} , the spectrum of A_{e_-} consists only of simple eigenvalues λ_n with eigenvectors $(\varphi_n, \lambda_n \varphi_n)$. Moreover, Proposition 5.1 implies that this set of eigenvectors is a Riesz basis of X .

Lemma 5.8. *There exists a sequence of coefficients $(c_n) \in \ell^2(\mathbb{C})$ such that*

$$\psi^\infty(x, t) = \sum_{n \in \mathbb{N}} c_n e^{-\lambda_n t} \varphi_n(x) .$$

Moreover, if μ is the real number defined in Proposition 5.4,

$$\mu = -\sup\{\operatorname{Re}(\lambda_n) / c_n \neq 0\} ,$$

and we have that, for all $\eta > 0$,

$$\psi^\infty(x, t) = o(e^{(\mu-\eta)t}) \quad \text{and} \quad e^{(\mu+\eta)t} = o(\psi^\infty(x, t)) , \quad (5.11)$$

when t goes to $-\infty$.

Proof : The set $(\varphi_n, \lambda_n \varphi_n)_{n \in \mathbb{N}}$ is a Riesz basis of X . As the eigenvalues are simple, from (5.4) we deduce that the associated biorthonormalised basis consists only of eigenvectors of A^* . We can assume that the eigenfunctions $(\varphi_n, \lambda_n \varphi_n)_{n \in \mathbb{N}}$ are conveniently normalized so that $(\bar{\varphi}_n, -\bar{\lambda}_n \bar{\varphi}_n)_{n \in \mathbb{N}}$ is the associated biorthonormalised basis. We recall that this biorthonormalised basis is also a Riesz basis of X (see [14]). So there exists $(d_n) \in \ell^2(\mathbb{C})$ such that

$$\begin{pmatrix} \psi^\infty \\ \psi_t^\infty \end{pmatrix} (0) = \sum_{n \in \mathbb{N}} d_n \begin{pmatrix} \bar{\varphi}_n \\ -\bar{\lambda}_n \bar{\varphi}_n \end{pmatrix} .$$

More precisely

$$d_n = \left\langle \begin{pmatrix} \psi^\infty \\ \psi_t^\infty \end{pmatrix} (0) \left| \begin{pmatrix} \varphi_n \\ \lambda_n \varphi_n \end{pmatrix} \right\rangle_X .$$

As $(\bar{\varphi}_n, -\bar{\lambda}_n \bar{\varphi}_n)$ is an eigenvector of

$$-\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A_{e^-} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

for the eigenvalue $-\bar{\lambda}_n$, we have

$$\begin{pmatrix} \psi^\infty \\ \psi_t^\infty \end{pmatrix} (t) = \sum_{n \in \mathbb{N}} d_n e^{-\bar{\lambda}_n t} \begin{pmatrix} \bar{\varphi}_n \\ -\bar{\lambda}_n \bar{\varphi}_n \end{pmatrix} . \quad (5.12)$$

As $D(B^{1/2})$ is continuously imbedded in $\mathcal{C}^0([0, 1])$, we can write

$$\psi^\infty(x, t) = \sum_{n \in \mathbb{N}} d_n e^{-\bar{\lambda}_n t} \bar{\varphi}_n(x) . \quad (5.13)$$

Since the spectrum of A_{e_-} is symmetric with respect to the real axis, the set of eigenpairs $\{(\lambda_n, \phi_n)\}$ is equal to the set $\{(\bar{\lambda}_n, \bar{\phi}_n)\}$. So, we can reorder the sum (5.13) to obtain a set of coefficients $\{c_n\}_{n \in \mathbb{N}} \in \ell^2(\mathbb{C})$ such that

$$\psi^\infty(x, t) = \sum_{n \in \mathbb{N}} c_n e^{-\lambda_n t} \phi_n(x) .$$

As μ is defined by

$$\mu = \lim_{t \rightarrow -\infty} \ln \left\| \begin{pmatrix} \psi^\infty \\ \psi_t^\infty \end{pmatrix} \right\|_X^{1/t} ,$$

the decomposition (5.12) implies that

$$\mu = -\sup\{\operatorname{Re}(\bar{\lambda}_n) / d_n \neq 0\} = -\sup\{\operatorname{Re}(\lambda_n) / c_n \neq 0\} .$$

Finally, the last claim of the lemma will directly follow from Proposition 5.5, if we prove that $(d_n \bar{\varphi}_n(x))_{n \in \mathbb{N}}$ belongs to $\ell^1(\mathbb{C})$.

Due to Proposition 5.4, $(\psi^\infty, \psi_t^\infty) \in D(A^*)$. So we can write in X

$$\begin{aligned} A^* \begin{pmatrix} \psi^\infty \\ \psi_t^\infty \end{pmatrix} (0) &= \sum \left\langle A^* \begin{pmatrix} \psi^\infty \\ \psi_t^\infty \end{pmatrix} (0) \left| \begin{pmatrix} \varphi_n \\ \lambda_n \varphi_n \end{pmatrix} \right\rangle_X \begin{pmatrix} \bar{\varphi}_n \\ -\bar{\lambda}_n \bar{\varphi}_n \end{pmatrix} \right. \\ &= \sum \left\langle \begin{pmatrix} \psi^\infty \\ \psi_t^\infty \end{pmatrix} (0) \left| A \begin{pmatrix} \varphi_n \\ \lambda_n \varphi_n \end{pmatrix} \right\rangle_X \begin{pmatrix} \bar{\varphi}_n \\ -\bar{\lambda}_n \bar{\varphi}_n \end{pmatrix} \right. \\ &= \sum d_n \bar{\lambda}_n \begin{pmatrix} \bar{\varphi}_n \\ -\bar{\lambda}_n \bar{\varphi}_n \end{pmatrix} . \end{aligned}$$

which implies that $|d_n \bar{\lambda}_n|^2$ is summable since $((\bar{\varphi}_n, -\bar{\lambda}_n \bar{\varphi}_n))_{n \in \mathbb{N}}$ is a Riesz basis. In addition, applying the equivalence of norms (2.8) to the vector $(\bar{\varphi}_n, -\bar{\lambda}_n \bar{\varphi}_n)$, we have that $\|(\bar{\varphi}_n, -\bar{\lambda}_n \bar{\varphi}_n)\|_X$ is uniformly bounded by $\frac{1}{a_1}$. So, we can write

$$\sum |d_n \bar{\varphi}_n(x)| \leq \frac{1}{a_1} \sum |d_n| \leq \frac{1}{a_1} \sqrt{\sum \frac{1}{|\lambda_n|^2}} \sqrt{\sum |\lambda_n d_n|^2} .$$

As we know, by Proposition 5.1, that Hypothesis (Spec)b) is also valid for A_{e_-} , we have that $\sum (\frac{1}{|\lambda_n|^2})$ is convergent, and thus $(d_n \bar{\varphi}_n(x))_{n \in \mathbb{N}}$ belongs to $\ell^1(\mathbb{C})$. \square

Fourth step : Conclusion

Summarizing the above arguments and computations, we get

$$\begin{aligned}
J &= J_0 + \sum_{i=1}^m J_i \\
&= \frac{\psi^\infty(t_\zeta)}{\lambda\phi(x)(\zeta - e_-(x))} + O\left(\frac{\psi^\infty(t_\zeta)}{(\zeta - e_-(x))^{1-\frac{\delta}{\lambda}}} + (\zeta - e_-(x))^{\frac{\mu}{\lambda}-1+\frac{\delta}{\lambda}}\right) \\
&\quad + S((\zeta - e_-(x))^r) + \omega(\zeta, \varepsilon) \\
&= G(\zeta) + \omega(\zeta, \varepsilon),
\end{aligned}$$

where $\omega(\zeta, \varepsilon) \rightarrow 0$ when $\varepsilon \rightarrow 0$, r is a rational number, and

$$t_\zeta = \frac{1}{\lambda} \ln \left[\frac{\zeta - e_-(x)}{b\varphi(x)} + O((\zeta - e_-(x))^2) \right]. \quad (5.14)$$

To prove that Hypothesis (h7)(c) of Theorem A.1 is satisfied, and so to complete our proof, we must find ζ and ε such that

$$J = G(\zeta) + \omega(\zeta, \varepsilon) \neq 0.$$

As $\omega(\zeta, \varepsilon) \rightarrow 0$ when ε goes to 0, we only need to find ζ such that $G(\zeta) \neq 0$, and then choose a positive number ε small enough to ensure that $G(\zeta) + \omega(\zeta, \varepsilon) \neq 0$. Assume that the first term of the series $S(z)$ is of order k (where k may be $+\infty$ if $S(z) = 0$). There are three cases.

- 1) If $kr < \frac{\mu}{\lambda} - 1$, then, using (5.14) and (5.11), we have that, for all $\eta > 0$,

$$\psi^\infty(t_\zeta) = o(\zeta - e_-(x))^{\frac{\mu}{\lambda}-\eta},$$

and in particular,

$$\frac{\psi^\infty(t_\zeta)}{\zeta - e_-(x)} = o(\zeta - e_-(x))^{kr}.$$

The dominant terms of J_0 and $\sum_{i=1}^m J_i$ are different, and so we can find ζ as small as needed such that $G(\zeta) \neq 0$.

- 2) Assume now that $kr > \frac{\mu}{\lambda} - 1$, we can conclude just as in the preceding case since we have

$$(\zeta - e_-(x))^{kr} = o\left(\frac{\psi^\infty(t_\zeta)}{\zeta - e_-(x)}\right).$$

3) Finally, we assume that $kr = \frac{\mu}{\lambda} - 1$. Notice that, by construction, \hat{f} is assumed to satisfy the irrational ratio property of Theorem 4.4. As $\frac{\mu}{\lambda} = kr + 1$ is a rational number, $-\mu$ cannot be a real negative eigenvalue of A_{e_-} or the number $-C_{ev}$. We know, applying Lemma 5.1, that A_{e_-} satisfies Hypothesis (Spec)b). Using Lemma 5.8, as

$$\mu = -\sup\{\operatorname{Re}(\lambda_n) / c_n \neq 0\} ,$$

the only possibility is that $-\mu$ is the real part of nonreal eigenvalues. Thus, there exists a finite number of nonreal eigenvalues $\lambda_1, \dots, \lambda_p$, with $\operatorname{Re}(\lambda_l) = -\mu$, such that

$$\psi^\infty(t) = \sum_{l=1}^p d_l e^{-\lambda_l t} + \sum_{\operatorname{Re}\lambda < -\mu} c_\lambda e^{-\lambda t} \phi_\lambda(x) ,$$

where the coefficients d_l are not zero. Since $-C_{ev}$ is the only accumulation point of the real part of the spectrum of A_{e_-} and $\mu \neq -C_{ev}$, we deduce from this equality that

$$\psi^\infty(t) = e^{\mu t} P(t) + o(e^{\mu t}) ,$$

where

$$P(t) = \sum_{l=1}^p d_l e^{-i\operatorname{Im}(\lambda_l)t} .$$

So we have,

$$\frac{\psi^\infty(t_\zeta)}{\lambda\phi(x)(\zeta - e_-(x))} = \frac{(\zeta - e_-(x))^{\frac{\mu}{\lambda}-1}}{\lambda\phi(x)} P(t_\zeta) + o(\zeta - e_-(x))^{\frac{\mu}{\lambda}-1} .$$

Let Cz^k be the dominant term of the series $S(z)$. We must prove, as in the other cases, that we can find ζ as small as needed such that

$$G(\zeta) = \left(C - \frac{P(t_\zeta)}{\lambda\phi(x)}\right)(\zeta - e_-(x))^{kr} + o(\zeta - e_-(x))^{kr} \neq 0 .$$

As $P(t)$ is a non-constant almost-periodic function, we can find a constant $C' \neq C$ and a sequence of times $(t_n) \rightarrow +\infty$ such that $P(t_n) = C'\lambda\phi(x)$. So, we have a sequence $(\zeta_n) \rightarrow e_-(x)$ with

$$G(\zeta_n) = (C - C')(\zeta_n - e_-(x))^{kr} + o(\zeta_n - e_-(x))^{kr} \neq 0 ,$$

and obviously, for n large enough, $G(\zeta_n) \neq 0$.

We have proved that, in all the cases, we can find ζ and ε small enough, such that

$$J = \int_{\mathbb{R}} \psi(x, t) g_{\zeta, \varepsilon}(u(x, t)) dt \neq 0.$$

Our proof is now complete.

A Appendix

A.1 Brunovský-Poláčik-Raugel Theorem

In this subsection, we recall Theorem 4.8 of [6]. This abstract theorem is the key point of [6], as it gives the genericity of the transversality of the heteroclinic orbits. Our Theorem 2.6 is a more concrete result but concerns only damped wave equations and so it is less general than the Brunovský-Poláčik-Raugel abstract theorem. Since we refer to it to prove Theorem 2.6, we recall Theorem 4.8 of [6]. Notice that the version of the Brunovský-Poláčik-Raugel theorem, that we recall here, is not exactly the one which can be found in [6], but is a slightly stronger version given in [7].

We recall that $\text{Ind}(E)$ denotes the Morse index of a hyperbolic equilibrium E , that is the dimension of the local unstable manifold of E .

We consider the abstract semilinear equation with a nonlinearity F depending on a parameter $\lambda \in \Lambda$, where Λ is a Banach space,

$$\frac{\partial U}{\partial t}(t) = AU(t) + F(U(t), \lambda), \quad t > 0, \quad U(0) = U_0. \quad (\text{A.1})$$

We also introduce a Banach space Z , with $X \subset Z$. We assume that

(FP) X is a reflexive Banach space and the inclusion of X into Z is continuous.

(AP) A is the generator of a \mathcal{C}^0 -semigroup on the Banach space X .

For $M > 0$ fixed, we introduce the open set $G = \{v \in X \mid \|v\|_Z < M\}$ in X and a mapping $F \in \mathcal{C}_b^r(G \times L, X)$, $r \geq 1$, where L is open in the Banach space Λ . The mapping $F(\cdot, \lambda) : x \in G \mapsto F(x, \lambda) \in X$ is of class $\mathcal{C}_{loc}^{1,1}(X, X)$ uniformly in $\lambda \in L$.

(BUP1) If $\lambda \in L$ and $U_1(t)$ and $U_2(t)$ are two solutions in $\mathcal{C}^0([0, T], X)$ of (A.1), and if there exists τ , $0 \leq \tau \leq T$ such that $U_1(\tau) = U_2(\tau)$, then $U_1(t) = U_2(t)$ for all $t \in [0, T]$.

(BUP2) If $\lambda \in L$ and $\tilde{U}(t)$ is a solution of (A.1) on an interval (t_1, t_2) , then the evolution operator $T_{\tilde{U}}(t, s) \in L(X, X)$ defined by the linear variational equation

$$\frac{\partial Y}{\partial t}(t) = AY(t) + DF(\tilde{U}(t), \lambda)Y(t) , \quad t > s , \quad Y(s) = Y_0 . \quad (\text{A.2})$$

is injective and its image is dense in X for any $t_1 < s \leq t < t_2$.

Theorem A.1. *Assume that (AP), (FP), (BUP1) and (BUP2), together with the following additional assumptions are satisfied.*

(h1) *The Banach space Λ is separable.*

(h2) *A has a compact resolvent.*

(h3) *For any bounded set $L_0 \subset L$, F belongs to the space of C^2 -functions of $G \times L_0$ into X whose derivatives up to order 2 are bounded on $G \times L_0$.*

(h4) *For any $\lambda \in L$, all equilibria of (A.1) are hyperbolic.*

(h5) *For any $\lambda \in L$, all nonconstant bounded (in the norm of X) solutions on \mathbb{R} of (A.1) are heteroclinic orbits.*

(h6) *For any $\lambda_0 \in L$ and any $R > 0$, there exist a neighborhood \mathcal{V}_0 of λ_0 in L and a positive constant $C = C(\lambda_0, R)$ such that, if $U(t)$ is a heteroclinic orbit of (A.1) for $\lambda \in \mathcal{V}_0$ and if $\max_{t \in \mathbb{R}} \|U(t)\|_X \leq R$, then $U(t)$ is a classical solution of (A.1) and*

$$\|U(t)\|_{C_b^0(\mathbb{R}, D(A)) \cap C_b^1(\mathbb{R}, X)} \leq C(\lambda_0, R) .$$

(h7) *Given any $\lambda_0 \in L$ and any neighborhood \mathcal{V} of λ_0 in L , there exist $\hat{\lambda} \in \mathcal{V}$ and a Banach space $\hat{\Lambda}$ with the following properties*

(a) *$\hat{\lambda} \in \hat{\Lambda}$ and $\hat{\Lambda}$ is continuously embedded in Λ .*

(b) *$\hat{\lambda}$ has an open neighborhood $\hat{\mathcal{V}}$ in $\hat{\Lambda}$ such that $\hat{\mathcal{V}} \subset \mathcal{V}$ and*

$$F|_{G \times \hat{\mathcal{V}}} \in C^r(G \times \hat{\mathcal{V}}, X) \text{ with the derivatives up to order } r \text{ bounded} ,$$

where $r > \text{Ind}(E) + 1$ for any equilibrium point E of (A.1) with $\lambda = \hat{\lambda}$.

(c) *If \tilde{U} is a heteroclinic solution of (A.1) with $\lambda = \hat{\lambda}$ and $\Psi(t)$, $t \in \mathbb{R}$, is a nontrivial bounded mild solution of*

$$\frac{\partial \Psi}{\partial t}(s) = -(A^* + DF^*(\tilde{U}(s), \hat{\lambda}))\Psi(s) , \quad s < t , \quad \Psi(t) = \Psi_0 .$$

then there exists $\lambda \in \hat{\Lambda}$ such that

$$\int_{-\infty}^{+\infty} \langle \Psi(t), D_\lambda F(U(t), \hat{\lambda})\lambda \rangle_X dt \neq 0 .$$

Under these assumptions, there is a generic (or residual) subset $\mathcal{L} \subset L$ such that for any $\lambda \in \mathcal{L}$, any heteroclinic orbit of (A.1) contained in G is transverse.

A.2 Sard-Smale theorem

The following theorem is a main tool to prove genericity results. We give here the simplest version (see for example [24] for a proof or stronger versions).

We recall that, if f is a differentiable function from X into Z , a value $z \in Z$ is said to be regular for f if for any $x \in f^{-1}(z)$, the differential $Df(x)$ is surjective.

Theorem A.2. *Let X, Y, Z be three Banach spaces, $U \subset X$ and $V \subset Y$ two open sets, and $\Phi : U \times V \longrightarrow Z$ be a mapping of class \mathcal{C}^r ($r \geq 1$). Let z be an element of Z . Assume that the following hypotheses hold :*

- (i) for each $(x, y) \in \Phi^{-1}(z)$, $D_x\Phi(x, y)$ is a Fredholm operator of index strictly less than r .*
- (ii) for each $(x, y) \in \Phi^{-1}(z)$, $D\Phi(x, y)$ is surjective.*
- (iii) X and Y are separable metric spaces.*

Then the set $\{y \in Y/z \text{ is a regular value of } \Phi(., y)\}$ is a generic subset of Y .

References

- [1] S.B. Angenent, *The Morse-Smale property for a semilinear parabolic equation*, Journal of Differential Equations n°62 (1986), pp. 427-442.
- [2] C. Bardos, G. Lebeau and J. Rauch, *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary*, SIAM Journal on Control and Optimization Vol 30 (1992), pp. 1024-1065.
- [3] R.A. Bonic, *Linear functional analysis*, Gordon et Breach science publishers (1969).
- [4] P. Brunovský and S.-N. Chow, *Generic properties of stationary state solutions of reaction-diffusion equations*, Journal of Differential Equations vol 53 (1984), pp. 1-23.
- [5] P. Brunovský and P. Poláčik, *The Morse-Smale structure of a generic reaction-diffusion equation in higher space dimension*, Journal of Differential Equations vol 135 (1997), pp. 129-181.
- [6] P. Brunovský and G. Raugel, *Genericity of the Morse-Smale property for damped wave equations*, Journal of Dynamics and Differential Equations vol 15 n°2 (2003), pp. 571-658.
- [7] P. Brunovský and G. Raugel, *Généricité de la propriété de Morse-Smale pour des équations hyperboliques avec dissipation*, Actes du Colloque franco-tunisien, Hammamet (2003), to appear in Astérisque.
- [8] G. Chen, S. A. Fulling, F. J. Narcowich and S. Sun *Exponential decay of energy of evolution equations with locally distributed damping*, SIAM Journal on Applied Mathematics vol 51 (1991), pp. 266–301.
- [9] I. Chueshov, M. Eller and I. Lasiecka, *On the attractor for a semilinear wave equation with critical exponent and nonlinear boundary dissipation*, Communications in Partial Differential Equations vol 27 (2002), pp. 1901–1951.
- [10] F. Conrad and Ö. Morgül, *On the stabilization of a flexible beam with a tip mass*, SIAM Journal on Control and Optimization vol 36 n°6 (1998), pp. 1962-1986.
- [11] S. Cox and E. Zuazua, *The rate at which energy decays in a damped string*, Communication in Partial Differential Equations vol 19 (1994), pp. 213-243.
- [12] S. Cox and E. Zuazua, *The rate at which energy decays in a string damped at one end*, Indiana University Mathematics Journal vol 44 n°2 (1995), pp. 545-573.

- [13] E. Feireisl and E. Zuazua, *Global attractors for semilinear wave equations with locally distributed nonlinear damping and critical exponent*, Communications in Partial Differential Equations vol 18 n° 9-10 (1993), pp. 1539-1555.
- [14] I.C. Gohberg and M.G. Krein, *Introduction to the theory of linear nonselfadjoint operators*, Translations of mathematical monographs n°18 (1969), American Mathematical Society.
- [15] M. Golubitsky and V. Guillemin, *Stable mapping and their singularities*, Graduate Texts in Mathematics n°14 (1974) Springer Verlag.
- [16] B.Z. Guo and K.Y. Chan, *Riesz basis generation, eigenvalues distribution, and exponential stability for a Euler-Bernoulli beam with joint feedback control*, Revista Matemática Complutense n°14 (2001), pp 205-229.
- [17] J.K. Hale *Asymptotic behavior of dissipative systems*, Mathematical Surveys and Monographs n°25 (1988), American Mathematical Society.
- [18] J.K. Hale, L. Magalhães and W. Oliva, *An introduction to infinite dimensional dynamical systems*, Applied Mathematical Sciences n°47 (1984), Springer-Verlag. Second edition (2002), *Dynamics in infinite dimensions*.
- [19] J.K. Hale and G. Raugel, *Attractors for dissipative evolutionary equations*, International Conference on Differential Equations vol 1, 2 (1991) World Sci. Publishing, pp. 3-22.
- [20] J.K. Hale and G. Raugel, *Regularity, determining modes and Galerkin methods*, Journal de Mathématiques Pures et Appliquées vol 82 (2003), pp. 1075-1136.
- [21] A. Haraux, *Une remarque sur la stabilisation de certains systemes du deuxieme ordre en temps*, Portugaliae Mathematica vol 46 (1989), pp. 246-257.
- [22] D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics n°840 (1981), Springer-Verlag.
- [23] D. Henry, *Some infinite-dimensional Morse-Smale systems defined by parabolic partial differential equations*, Journal of Differential Equations vol 59 (1985), pp. 165-205.
- [24] D. Henry *Perturbation of the Boundary for Boundary Value Problems of Partial Differential Operators*, Cambridge University Press, Cambridge UK, to appear.
- [25] T. Kato, *Perturbation theory for linear operators*, Die Grundlehren der mathematischen Wissenschaften n°152 (1966), Springer-Verlag.

- [26] V. Komornik and E. Zuazua, *A direct method for the boundary stabilization of the wave equation*, Journal de Mathématiques Pures et Appliquées vol 69 (1990), pp 33-55.
- [27] I.N. Kostin, *Attractor for a semilinear wave equation with boundary damping*, Journal of Mathematical Sciences vol 98 (2000), pp. 753-764.
- [28] I. Lasiecka and D. Tataru, *Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping*, Differential and Integral Equations vol 6 (1993), pp 507-533.
- [29] S. Li, J. Yu, Z. Liang and G. Zhu, *Stabilization of high eigenfrequencies of a beam equation with generalized viscous damping*, SIAM Journal on Control and Optimization vol 37 n°6 (1999), pp. 1767-1779.
- [30] P.J. McKenna and I. Lasiecka, *Some perturbation results and their applications to stabilization of hyperbolic systems*, SIAM Journal of Mathematical Analysis n°15 (1984), pp. 1140-1152.
- [31] C. Miranda, *Partial differential equations of elliptic type* (1970), Springer-Verlag.
- [32] J. Palis and W. De Melo, *Geometric theory of dynamical systems* (1982), Springer-Verlag.
- [33] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences n°44 (1983), Springer-Verlag.
- [34] B. Rao, *Optimal decay rate in a damped Rayleigh beam*, Contemporary Mathematics vol 209 (1997), pp. 211-229.
- [35] J. Rauch, *Qualitative behavior of dissipative wave equations on bounded domains*, Archive for Rational Mechanics and Analysis vol 62 (1976), pp. 77-85.
- [36] M. Shubov, *Riesz basis property of root vectors of non-selfadjoint operators generated by radial damped wave equation*, Advances in Differential Equations vol 5 (2000), pp. 623-656.
- [37] D. Tataru, *Uniform decay rates and attractors for evolution PDE's with boundary dissipation*, Journal of Differential Equations vol 121 (1995), pp. 1-27.
- [38] E. Zuazua, *Exponential decay for the semilinear wave equation with locally distributed damping* Communications in Partial Differential Equations vol 15 n°2 (1990), pp. 205-235.
- [39] E. Zuazua, *Uniform stabilization of the wave equation by nonlinear boundary feedback*, SIAM Journal on Control and Optimization vol 28 n°2 (1990), pp. 466-477.