Global stability of travelling fronts for a damped wave equation with bistable nonlinearity

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Abstract: We consider the damped wave equation $\alpha u_{tt} + u_t = u_{xx} - V'(u)$ on the whole real line, where V is a bistable potential. This equation has travelling front solutions of the form $u(x,t) = h(x - st)$ which describe a moving interface between two different steady states of the system, one of which being the global minimum of V. We show that, if the initial data are sufficiently close to the profile of a front for large $|x|$, the solution of the damped wave equation converges uniformly on $\mathbb{R}$ to a travelling front as $t \to +\infty$. The proof of this global stability result is inspired by a recent work of E. Risler [38] and relies on the fact that our system has a Lyapunov function in any Galilean frame.

Keywords: Travelling wave, global stability, damped wave equation, Lyapunov function.

Codes AMS (2000) : 35B35, 35B40, 37L15, 37L70.

1 Introduction

The aim of this paper is to describe the long-time behavior of a large class of solutions of the semilinear damped wave equation

$$\alpha u_{tt} + u_t \;=\; u_{xx} - V'(u) \;, \tag{1.1}$$

where $\alpha > 0$ is a parameter, $V : \mathbb{R} \to \mathbb{R}$ is a smooth bistable potential, and the unknown $u = u(x,t)$ is a real-valued function of $x \in \mathbb{R}$ and $t \ge 0$. Equations of this form appear in many different contexts, especially in physics and in biology. For instance, Eq. (1.1) describes the continuum limit of an infinite chain of coupled oscillators, the propagation of voltage along a nonlinear transmission line [4], and the evolution of an interacting population if the spatial spread of the individuals is modelled by a velocity jump process instead of the usual Brownian motion [18, 21, 24].

As was already observed by several authors, the long-time asymptotics of the solutions of the damped wave equation (1.1) are quite similar to those of the corresponding reaction-diffusion equation $u_t = u_{xx} - V'(u)$. In particular, if $V'(u)$ vanishes rapidly enough as $u \to 0$, the solutions of (1.1) originating from small and localized initial data converge as $t \to +\infty$ to the same self-similar profiles as in the parabolic case [12, 23, 27, 34, 35]. The analogy persists for solutions with nontrivial limits as $x \to \pm\infty$, in which case the long-time asymptotics are often described by uniformly translating solutions of the form $u(x,t) = h(x - st)$, which are usually called *travelling fronts*. Existence of such solutions for hyperbolic equations of the form (1.1) was first proved by Hadeler [19, 20], and a few stability results were subsequently obtained by Gallay & Raugel [10, 11, 13, 14].

While local stability is an important theoretical issue, in the applications one is often interested in *global convergence results* which ensure that, for a large class of initial data with a prescribed behavior at infinity, the solutions approach travelling fronts as $t \to +\infty$. For the scalar parabolic equation $u_t = u_{xx} - V'(u)$, such results were obtained by Kolmogorov, Petrovski & Piskunov [29], by Kanel [25, 26], and by Fife & McLeod [8, 9] under various assumptions on the potential. All the proofs use in an essential way comparison theorems based on the maximum principle. These techniques are very powerful to obtain global information on the solutions, and were also successfully applied to monotone parabolic systems [44, 41] and to parabolic equations on infinite cylinders [39, 40].

However, unlike its parabolic counterpart, the damped wave equation (1.1) has no maximum principle in general. More precisely, solutions of (1.1) taking their values in some interval $I \subset \mathbb{R}$ obey a comparison principle only if

$$4\alpha \sup_{u \in I} V''(u) \;\le\; 1 \;, \tag{1.2}$$

see [37] or [11, Appendix A]. In physical terms, this condition means that the relaxation time α is small compared to the period of the nonlinear oscillations. In particular, if I is a neighborhood of a local minimum $\bar{u}$ of V, inequality (1.2) implies that the linear oscillator $\alpha u_{tt} + u_t + V''(\bar{u})u = 0$ is strongly damped, so that no oscillations occur. It was shown in [11, 13] that the travelling fronts of (1.1) with a monostable nonlinearity are stable against large perturbations provided that the parameter α is sufficiently small so that the strong damping condition (1.2) holds for the solutions under consideration. In other words, the basin of attraction of the hyperbolic travelling fronts becomes arbitrarily large as $\alpha \to 0$, but if α is not assumed to be small there is no hope to use "parabolic" methods to obtain global stability results for the travelling fronts of the damped wave equation (1.1).

Recently, however, a different approach to the stability of travelling fronts has been developped by Risler [15, 38]. The new method is purely variational and is therefore restricted to systems that possess a gradient structure, but its main interest lies in the fact that it does not rely on the maximum principle. The power of this approach is demonstrated in the pioneering work [38] where global convergence results are obtained for the non-monotone reaction-diffusion system $u_t = u_{xx} - \nabla V(u)$, with $u \in \mathbb{R}^n$ and $V : \mathbb{R}^n \to \mathbb{R}$. The aim of the present article is to show that Risler's method can be adapted to the damped hyperbolic equation (1.1) and allows in this context to prove global convergence results *without any smallness assumption* on the parameter α.

Before stating our theorem, we need to specify the assumptions we make on the non-linearity in (1.1). We suppose that $V \in \mathcal{C}^3(\mathbb{R})$, and that there exist positive constants a and b such that

$$uV'(u) \geq au^2 - b\,, \quad \text{for all } u \in \mathbb{R}\,. \tag{1.3}$$

In particular, $V(u) \to +\infty$ as $|u| \to \infty$. We also assume

$$V(0) = 0\,, \quad V'(0) = 0\,, \quad V''(0) > 0\,, \tag{1.4}$$

$$V(1) < 0\,, \quad V'(1) = 0\,, \quad V''(1) > 0\,. \tag{1.5}$$

Finally we suppose that, except for $V(0)$ and $V(1)$, all critical values of V are positive:

$$\Big\{u \in \mathbb{R} \,\Big|\, V'(u) = 0\,,\ V(u) \leq 0\Big\} = \{0\,;\,1\}\,. \tag{1.6}$$

In other words V is a smooth, strictly coercive function which reaches its global minimum at $u = 1$ and has in addition a local minimum at $u = 0$. We call V a *bistable* potential because both $u = 0$ and $u = 1$ are stable equilibria of the one-dimensional dynamical system $\dot{u} = -V'(u)$. The simplest example of such a potential is represented in Fig. 1. Note however that V is allowed to have positive critical values, including local minima.

Under assumptions (1.4)–(1.6), it is well-known that the parabolic equation $u_t = u_{xx} - V'(u)$ has a family of travelling fronts of the form $u(x,t) = h(x - c_* t - x_0)$ connecting the stable equilibria $u = 1$ and $u = 0$, see e.g. [2]. More precisely, there exists a unique speed $c_* > 0$ such that the boundary value problem

$$\begin{cases} h''(y) + c_* h'(y) - V'(h(y)) = 0\,, \quad y \in \mathbb{R}\,, \\ h(-\infty) = 1\,, \quad h(+\infty) = 0\,, \end{cases} \tag{1.7}$$

has a solution $h : \mathbb{R} \to (0,1)$, in which case the profile h itself is unique up to a translation. Moreover $h \in \mathcal{C}^4(\mathbb{R})$, $h'(y) < 0$ for all $y \in \mathbb{R}$, and $h(y)$ converges exponentially toward its limits as $y \to \pm\infty$. As was observed in [11, 19], for any $\alpha > 0$ the damped hyperbolic equation (1.1) has a corresponding family of travelling fronts given by

$$u(x,t) = h(\sqrt{1 + \alpha c_*^2}\, x - c_* t - x_0)\,, \quad x_0 \in \mathbb{R}\,. \tag{1.8}$$

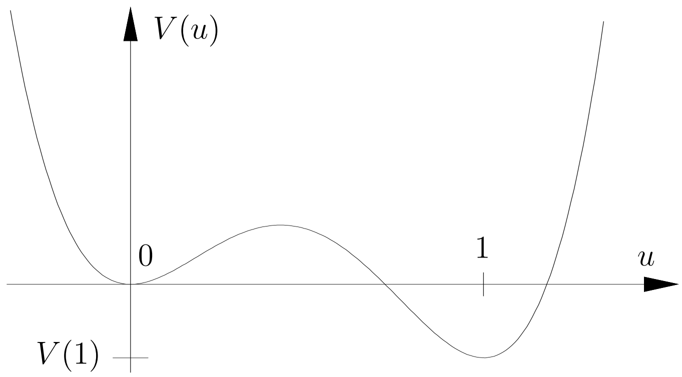

Fig. 1: The simplest example of a potential V satisfying assumptions (1.3)–(1.6).

Remark that the actual speed of these waves is not c_*, but $s_* = c_*/\sqrt{1+\alpha c_*^2}$. In particular s_* is smaller than $1/\sqrt{\alpha}$ (the slope of the characteristics of Eq. (1.1)), which means that the travelling fronts (1.8) are always "subsonic". In what follows we shall refer to c_* as the "parabolic speed" to distinguish it from the physical speed s_*.

We are now in position to state our main result:

Theorem 1.1. *Let $\alpha > 0$ and let $V \in \mathcal{C}^3(\mathbb{R})$ satisfy (1.3)–(1.6) above. Then there exist positive constants δ and ν such that, for all initial data $(u_0, u_1) \in H^1_{\mathrm{ul}}(\mathbb{R}) \times L^2_{\mathrm{ul}}(\mathbb{R})$ satisfying*

$$\limsup_{\xi\to-\infty} \int_\xi^{\xi+1} \Big((u_0(x)-1)^2 + u_0'(x)^2 + u_1(x)^2\Big)\,\mathrm{d}x \;\le\; \delta\;, \tag{1.9}$$

$$\limsup_{\xi\to+\infty} \int_\xi^{\xi+1} \Big(u_0(x)^2 + u_0'(x)^2 + u_1(x)^2\Big)\,\mathrm{d}x \;\le\; \delta\;, \tag{1.10}$$

equation (1.1) has a unique global solution (for positive times) such that $u(\cdot,0) = u_0$, $u_t(\cdot,0) = u_1$. Moreover, there exists $x_0 \in \mathbb{R}$ such that

$$\sup_{x\in\mathbb{R}} \Big|u(x,t) - h(\sqrt{1+\alpha c_*^2}\,x - c_* t - x_0)\Big| \;=\; \mathcal{O}(e^{-\nu t})\;, \quad \textit{as } t \to +\infty\;. \tag{1.11}$$

Remarks:

1. Loosely speaking Theorem 1.1 says that, if the initial data (u_0, u_1) are close enough to the global equilibrium $(1,0)$ as $x \to -\infty$ and to the local equilibrium $(0,0)$ as $x \to +\infty$, the solution $u(x,t)$ of (1.1) converges uniformly in space and exponentially fast in time toward a member of the family of travelling fronts (1.8). In particular, any solution which looks roughly like a travelling front at initial time will eventually approach a suitable translate of that front. It should be noted, however, that our result does not give any constructive estimate of the time needed to reach the asymptotic regime described by (1.11). Depending on the shape of the potential and of the initial data, very long transients can occur before the solution actually converges to a travelling front.

2. The definition of the uniformly local Lebesgue space $L^2_{\mathrm{ul}}(\mathbb{R})$ and the uniformly local Sobolev space $H^1_{\mathrm{ul}}(\mathbb{R})$ will be recalled at the beginning of Section 2. These spaces provide a very convenient framework to study infinite-energy solutions of the hyperbolic equation (1.1), but their knowledge is not necessary to understand the meaning of Theorem 1.1. In a first reading one can assume, for instance, that u_0' and u_1 are bounded and uniformly continuous functions, in which case assumptions (1.9), (1.10) can be replaced by

$$\limsup_{x\to-\infty}(|u_0(x)-1|+|u_0'(x)|+|u_1(x)|) \,\le\, \delta\;, \quad \limsup_{x\to+\infty}(|u_0(x)|+|u_0'(x)|+|u_1(x)|) \,\le\, \delta\;.$$

Also, to simplify the presentation, we have expressed our convergence result (1.11) in the uniform norm, but the proof will show that the solution $u(x,t)$ of (1.1) converges to a travelling front in the uniformly local energy space $H^1_{\rm ul}(\mathbb{R}) \times L^2_{\rm ul}(\mathbb{R})$, see (9.5) below.

3. The convergence rate ν in (1.11) is related to the spectral gap of the linearization of (1.1) at the travelling front. As is shown in Section 9, we can take $\nu = \mathcal{O}(1)$ in the parabolic limit $\alpha \to 0$, whereas $\nu = \mathcal{O}(1/\alpha)$ as $\alpha \to +\infty$. On the other hand, the parameter δ in (1.9), (1.10) must be chosen small enough so that the following two conditions are satisfied. First, the initial data (u_0, u_1) should lie in the local basin of attraction of the steady state $(0,0)$ for large positive x, and in the basin of $(1,0)$ for large negative x. Next, the energy integral

$$\int_{-\infty}^0 e^{cx}\Big(\frac{\alpha}{2}u_1(x)^2 + \frac{1}{2}u_0'(x)^2 + V(u_0(x))\Big)\,{\rm d}x\;,$$

which is well-defined for any $c > 0$, should diverge to $-\infty$ as $c \to 0$. The second condition is an essential ingredient of our variational proof, but we do not know if the conclusion of Theorem 1.1 still holds without such an assumption.

The proof of Theorem 1.1 relies on the fact that Eq. (1.1) has, at least formally, a whole family of *Lyapunov functions*. To see this, let $u(x,t)$ be a solution of (1.1) whose initial data satisfy (1.9), (1.10). Given any $c \ge 0$ we go to a uniformly translating frame by setting

$$u(x,t) \,=\, u_c(\sqrt{1+\alpha c^2}\,x - ct\,,\,t)\;, \quad \text{or} \quad u_c(y,t) \,=\, u\Big(\frac{y+ct}{\sqrt{1+\alpha c^2}}\,,\,t\Big)\;. \tag{1.12}$$

The new function $u_c(y,t)$ is then a solution of the modified equation

$$\alpha\ddot{u}_c + \dot{u}_c - 2\alpha c\dot{u}_c' \,=\, u_c'' + cu_c' - V'(u_c)\;, \tag{1.13}$$

where $\dot{u}_c(y,t) \equiv \partial_t u_c(y,t)$ and $u_c'(y,t) \equiv \partial_y u_c(y,t)$. If we now introduce the energy function

$$E_c(t) \,=\, \int_{\mathbb{R}} e^{cy}\Big(\frac{\alpha}{2}(\dot{u}_c(y,t))^2 + \frac{1}{2}(u_c'(y,t))^2 + V(u_c(y,t))\Big)\,{\rm d}y\;, \tag{1.14}$$

a direct calculation shows that

$$E_c'(t) \,=\, -(1+\alpha c^2)\int_{\mathbb{R}} e^{cy}(\dot{u}_c(y,t))^2\,{\rm d}y \,\le\, 0\;. \tag{1.15}$$

In other words, Eq. (1.1) possesses (at least formally) a continuous family of non-equivalent Lyapunov functions, indexed by the parabolic speed $c \ge 0$. In the parabolic case $\alpha = 0$, it is shown in [15] that this rich Lyapunov structure is sufficient to prove the convergence (1.11) if we restrict ourselves to solutions which decay sufficiently rapidly to zero as $x \to +\infty$, and we believe that the approach of [15] works in the hyperbolic case too.

However, it is important to realize that the solutions we consider in Theorem 1.1 are only supposed to be small for large positive x, and do not necessarily converge to zero as $x \to +\infty$. Under these assumptions the integral in (1.14), which contains the exponentially growing factor e^{cy}, is usually divergent at $+\infty$, so that the Lyapunov function E_c is certainly not well-defined. This is a technical problem which seriously complicates the analysis. To overcome this difficulty, a possibility is to truncate the exponential factor e^{cy} in (1.14) to make it integrable over $\mathbb{R}$, see [8], [38]. We choose here another solution which consists in decomposing the solution $u(x,t)$ into a principal part $v(x,t)$ which is compactly supported to the right, and a small remainder $r(x,t)$ which decays exponentially to zero as $t \to +\infty$. The idea is then to study the approximate Lyapunov function defined by (1.14) with $u_c(x,t)$ replaced by $v_c(x,t)$, see Section 4 for more details.

As was already mentioned, the proof of Theorem 1.1 closely follows the previous work [38] which deals with gradient reaction-diffusion systems of the form $u_t = u_{xx} - \nabla V(u)$. There are, however, significant differences that we want to emphasize. First, the evolution defined by the damped hyperbolic equation (1.1) is not regularizing in finite time, but only asymptotically as $t \to +\infty$. As a consequence, the compactness arguments which play an essential role in the proof become slightly more delicate in the hyperbolic case. On the other hand, the solutions of (1.1) have a *finite speed of propagation*, a property which has no parabolic analog. Although this is not an essential ingredient of the proof, we shall take advantage of this fact here and there to get a priori estimates on the solutions of (1.1). Finally, an important property of the *scalar* equation (1.1) is that the associated elliptic problem (1.7) has a unique solution (h, c_*), and that the corresponding travelling front is a stable solution of (1.1). This is no longer true for the systems considered in [38], in which several stable or unstable fronts with different speeds may connect the same pair of equilibria. In this more general situation, without additional assumptions one can only show that the solution $u(x,t)$ approaches as $t \to \infty$ the family of all travelling fronts with a given speed.

Besides these natural differences due to the properties of Eq. (1.1), we also made technical choices in our proof which substantially differ from [38]. As was already mentioned, the most important one is that we give a meaning to the Lyapunov function E_c by decomposing the solution $u(x,t)$, and not by truncating the exponential weight e^{cy}. The main avantage of this approach is that the behavior of the energy is then easier to control. However, new arguments are required which have no counterpart in [38] or [15]. This is the case in particular of Section 6, which is the main technical step in our proof.

The rest of this paper is organized as follows. In Section 2, we briefly present the uniformly local spaces and we study the Cauchy problem for Eq. (1.1) in this framework. In Section 3, we prove the persistence of the boundary conditions (1.9), (1.10) and we decompose the solution of (1.1) as $u(x,t) = v(x,t) + r(x,t)$, where v is compactly supported to the right and r decays exponentially as $t \to +\infty$. We also introduce the invasion point $\overline{x}(t)$ which tracks the position of the moving interface. The core of the proof starts in Section 4, where we control the behavior of the energy E_c in a frame moving at constant speed $s = c/\sqrt{1 + \alpha c^2}$. These estimates are used in Section 5 to prove that the average speed $\overline{x}(t)/t$ converges to a limit $s_\infty \in (0, 1/\sqrt{\alpha})$ as $t \to +\infty$. The main technical step is Section 6, where we show that the energy stays uniformly bounded in a frame following the invasion point, see Proposition 6.1 for a precise statement. This allows us to prove in Section 7 that the solution $u(x,t)$ converges as $t \to +\infty$ to a travelling front uniformly in any interval of the form $(\overline{x}(t) - L, +\infty)$. The proof of Theorem 1.1 is then completed in two steps. In Section 8, we use an energy estimate in the laboratory frame to show that the solution $u(x,t)$ converges uniformly on $\mathbb{R}$ to a travelling front, at least for a sequence of times. Finally, the local stability result established in Section 9 gives the convergence for all times and the exponential rate in (1.11).

Notations. The symbols $K_0, K_1, \ldots$ denote our main constants, which will be used throughout the paper. In contrast, the local constants $C_0, C_1, \ldots$ will change from a section to another. We also denote by C a positive constant which may change from place to place.

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2 Global existence and asymptotic compactness

In this section, we prove that the Cauchy problem for Eq. (1.1) is globally well-posed for positive times in the uniformly local energy space $X = H^1_{\rm ul}(\mathbb{R}) \times L^2_{\rm ul}(\mathbb{R})$. We first recall the definitions of the uniformly local Sobolev spaces which provide a natural framework for the study of partial differential equations on unbounded domains, see e.g. [1, 5, 6, 16, 17, 28, 30, 31, 32].

For any $u \in L^2_{\rm loc}(\mathbb{R})$ we denote

$$\|u\|_{L^2_{\rm ul}} = \sup_{\xi\in\mathbb{R}} \Bigl(\int_\xi^{\xi+1} |u(x)|^2 \,\mathrm{d}x\Bigr)^{1/2} = \sup_{\xi\in\mathbb{R}} \|u\|_{L^2([\xi,\xi+1])} \le \infty\ . \tag{2.1}$$

The uniformly local Lebesgue space is defined as

$$L^2_{\rm ul}(\mathbb{R}) = \Bigl\{u \in L^2_{\rm loc}(\mathbb{R}) \Bigm| \|u\|_{L^2_{\rm ul}} < \infty\,,\ \lim_{\xi\to 0} \|T_\xi u - u\|_{L^2_{\rm ul}} = 0\Bigr\}\ , \tag{2.2}$$

where T_ξ denotes the translation operator: $(T_\xi u)(x) = u(x-\xi)$. In a similar way, for any $k \in \mathbb{N}$, we introduce the uniformly local Sobolev space

$$H^k_{\rm ul}(\mathbb{R}) = \Bigl\{u \in H^k_{\rm loc}(\mathbb{R}) \Bigm| \partial^j u \in L^2_{\rm ul}(\mathbb{R}) \text{ for } j = 0,1,2,\ldots,k\Bigr\}\ , \tag{2.3}$$

which is equipped with the natural norm

$$\|u\|_{H^k_{\rm ul}} = \Bigl(\sum_{j=0}^k \|\partial^j u\|^2_{L^2_{\rm ul}}\Bigr)^{1/2}\ .$$

It is easy to verify that $H^k_{\mathrm{ul}}(\mathbb{R})$ is a Banach space, which is however neither reflexive nor separable. If $\mathcal{C}^k_{\mathrm{bu}}(\mathbb{R})$ denotes the Banach space of all $u \in \mathcal{C}^k(\mathbb{R})$ such that $\partial^j u$ is bounded and uniformly continuous for $j = 0, \dots, k$, we have the continuous inclusions

$$\mathcal{C}^k_{\mathrm{bu}}(\mathbb{R}) \hookrightarrow H^k_{\mathrm{ul}}(\mathbb{R}) \hookrightarrow \mathcal{C}^{k-1}_{\mathrm{bu}}(\mathbb{R}) .$$

In particular $H^1_{\mathrm{ul}}(\mathbb{R}) \hookrightarrow \mathcal{C}^0_{\mathrm{bu}}(\mathbb{R})$ and $\|u\|^2_{L^\infty} \le 2\|u\|^2_{H^1_{\mathrm{ul}}}$ for all $u \in H^1_{\mathrm{ul}}(\mathbb{R})$. Note also that $H^k_{\mathrm{ul}}(\mathbb{R})$ is an algebra for any $k \ge 1$, i.e. $\|uv\|_{H^k_{\mathrm{ul}}} \le C\|u\|_{H^k_{\mathrm{ul}}}\|v\|_{H^k_{\mathrm{ul}}}$ for all $u, v \in H^k_{\mathrm{ul}}(\mathbb{R})$. Finally the space $\mathcal{C}^\infty_{\mathrm{bu}}(\mathbb{R})$ is dense in $H^k_{\mathrm{ul}}(\mathbb{R})$ for any $k \in \mathbb{N}$.

Remark: Some authors do not include in the definition of the uniformly local L^2 space the assumption that $\xi \mapsto T_\xi u$ is continuous for any $u \in L^2_{\mathrm{ul}}(\mathbb{R})$. The resulting uniformly local Sobolev spaces are of course larger, but also less convenient from a functional-analytic point of view. In particular, one looses the property that $H^{k+1}_{\mathrm{ul}}(\mathbb{R})$ is dense in $H^k_{\mathrm{ul}}(\mathbb{R})$. As we shall see, the definitions (2.2), (2.3) guarantee that the damped wave equation (1.1) defines a continuous evolution in $H^1_{\mathrm{ul}}(\mathbb{R}) \times L^2_{\mathrm{ul}}(\mathbb{R})$.

Let $X = H^1_{\mathrm{ul}}(\mathbb{R}) \times L^2_{\mathrm{ul}}(\mathbb{R})$ and $Y = H^2_{\mathrm{ul}}(\mathbb{R}) \times H^1_{\mathrm{ul}}(\mathbb{R})$. The main result of this section is:

Proposition 2.1. *For all initial data $(u_0, u_1) \in X$, Eq. (1.1) has a unique global solution $u \in \mathcal{C}^0([0,+\infty), H^1_{\mathrm{ul}}(\mathbb{R})) \cap \mathcal{C}^1([0,+\infty), L^2_{\mathrm{ul}}(\mathbb{R}))$ satisfying $u(0) = u_0$, $u_t(0) = u_1$. This solution depends continuously on the initial data, uniformly in time on compact intervals. Moreover, there exists $K_\infty > 0$ (depending only on α and V) such that*

$$\limsup_{t \to +\infty} \left(\|u(\cdot,t)\|^2_{H^1_{\mathrm{ul}}} + \|u_t(\cdot,t)\|^2_{L^2_{\mathrm{ul}}}\right) \le K_\infty . \tag{2.4}$$

Proof: Setting $w = (u, u_t)$, we rewrite (1.1) as a first order evolution equation

$$w_t = Aw + F(w) , \tag{2.5}$$

where

$$A = \frac{1}{\alpha}\begin{pmatrix} 0 & \alpha \\ \partial_x^2 - 1 & -1 \end{pmatrix} , \quad \text{and} \quad F(w) = \frac{1}{\alpha}\begin{pmatrix} 0 \\ u - V'(u) \end{pmatrix} . \tag{2.6}$$

Using d'Alembert's formula for the solution of the wave equation $\alpha u_{tt} = u_{xx}$, it is straightforward to verify that the linear operator A_0 on X defined by

$$D(A_0) = Y , \quad A_0 = \frac{1}{\alpha}\begin{pmatrix} 0 & \alpha \\ \partial_x^2 & 0 \end{pmatrix} ,$$

is the generator of a strongly continuous group of bounded linear operators in X. The same is true for the linear operator A, which is a bounded perturbation of A_0, see [36, Section 3.1]. On the other hand, as $V \in \mathcal{C}^3(\mathbb{R})$ and $H^1_{\mathrm{ul}}(\mathbb{R}) \hookrightarrow \mathcal{C}^0_{\mathrm{bu}}(\mathbb{R})$, it is clear that the nonlinearity F maps X into Y, and that F is Lipschitz continuous on any bounded set $B \subset X$. Thus a classical argument shows that the Cauchy problem for (2.5) is locally well-posed in X, see [36, Section 6.1] or [16, Section 7.2]. More precisely, for any $r > 0$, there exists $T(r) > 0$ such that, for all initial data $w_0 \in X$ with $\|w_0\|_X \le r$, Eq. (2.5) has a unique (mild) solution $w \in \mathcal{C}^0([0,T], X)$ satisfying $w(0) = w_0$. This solution $w(t)$

depends continuously on the initial data w_0 in X, uniformly for all $t \in [0,T]$. Moreover, if $w_0 \in Y$, then $w \in \mathcal{C}^1([0,T],X) \cap \mathcal{C}^0([0,T],Y)$ is a classical solution of (2.5). To prove Proposition 2.1, it remains to show that all solutions of (2.5) stay bounded for positive times (hence can be extended to global solutions), and are eventually contained in an attracting ball whose radius is independent of the initial data.

Assume that $w = (u, u_t) \in \mathcal{C}^0([0,T],X)$ is a solution of (2.5). Let $\rho(x) = \exp(-\kappa|x|)$, where $\kappa > 0$ is small enough so that $2\sqrt{\alpha}\kappa \le 1$ and $\kappa^2 \le a$, with $a > 0$ as in (1.3). For any $\xi \in \mathbb{R}$ and all $t \in [0,T]$, we define

$$\mathcal{E}(\xi,t) \;=\; \int_{\mathbb{R}} (T_\xi\rho)(x)\Big(\alpha^2 u_t^2 + \alpha u_x^2 + 2\alpha \overline{V}(u) + \frac{1}{2}u^2 + \alpha u u_t\Big)(x,t)\,\mathrm{d}x\;, \tag{2.7}$$

$$\mathcal{F}(\xi,t) \;=\; \int_{\mathbb{R}} (T_\xi\rho)(x)\Big(\alpha u_t^2 + u_x^2 + a u^2\Big)(x,t)\,\mathrm{d}x\;, \tag{2.8}$$

where $\overline{V}(u) = V(u) - V(1) \ge 0$ and $(T_\xi\rho)(x) = \rho(x-\xi)$. We also denote

$$M(t) \;=\; \sup_{\xi\in\mathbb{R}} \mathcal{E}(\xi,t)\;, \quad t \in [0,T]\;. \tag{2.9}$$

Since $u(\cdot,t) \in H^1_{\mathrm{ul}}(\mathbb{R})$ and $u_t(\cdot,t) \in L^2_{\mathrm{ul}}(\mathbb{R})$, it is clear that $M(t) < \infty$ for all $t \in [0,T]$. Moreover, as $\overline{V}(u) \ge 0$ and $|\alpha u u_t| \le \frac{3\alpha^2}{4}u_t^2 + \frac{1}{3}u^2$, we have

$$\mathcal{E}(\xi,t) \;\ge\; \int_{\mathbb{R}} (T_\xi\rho)(x)\Big(\frac{\alpha^2}{4}u_t^2 + \alpha u_x^2 + \frac{1}{6}u^2\Big)(x,t)\,\mathrm{d}x\;.$$

Taking in both sides the supremum over $\xi \in \mathbb{R}$ and using the definitions (2.1)–(2.3), we see that there exists $C_1 > 0$ (depending only on α) such that

$$\|w(\cdot,t)\|_X^2 \equiv \|u(\cdot,t)\|_{H^1_{\mathrm{ul}}}^2 + \|u_t(\cdot,t)\|_{L^2_{\mathrm{ul}}}^2 \le C_1 M(t)\;.$$

On the other hand, differentiating $\mathcal{E}(\xi,t)$ with respect to time, we find

$$\partial_t \mathcal{E}(\xi,t) \;=\; -\int_{\mathbb{R}} (T_\xi\rho)(\alpha u_t^2 + u_x^2 + uV'(u))\,\mathrm{d}x - \int_{\mathbb{R}} (T_\xi\rho)'(uu_x + 2\alpha u_x u_t)\,\mathrm{d}x\;.$$

To estimate the last integral, we observe that

$$-\int_{\mathbb{R}} (T_\xi\rho)' u u_x\,\mathrm{d}x \;=\; \frac{\kappa^2}{2}\int_{\mathbb{R}} (T_\xi\rho)u^2\,\mathrm{d}x - \kappa u(\xi)^2 \;\le\; \frac{\kappa^2}{2}\int_{\mathbb{R}} (T_\xi\rho)u^2\,\mathrm{d}x\;,$$

and

$$\Big|\int_{\mathbb{R}} (T_\xi\rho)' 2\alpha u_x u_t\,\mathrm{d}x\Big| \;\le\; 2\alpha\kappa\int_{\mathbb{R}} (T_\xi\rho)|u_x u_t|\,\mathrm{d}x \;\le\; \sqrt{\alpha}\kappa \int_{\mathbb{R}} (T_\xi\rho)(\alpha u_t^2 + u_x^2)\,\mathrm{d}x\;.$$

Using (1.3) together with our assumptions on κ, we arrive at

$$\partial_t \mathcal{E}(\xi,t) \;\le\; -\frac{1}{2}\int_{\mathbb{R}} (T_\xi\rho)(\alpha u_t^2 + u_x^2 + a u^2)\,\mathrm{d}x + \frac{2b}{\kappa} \;=\; -\frac{1}{2}\mathcal{F}(\xi,t) + \frac{2b}{\kappa}\;. \tag{2.10}$$

This differential inequality implies that the quantity $M(t)$ defined in (2.9) is a decreasing function of time as long as it stays above a certain threshold. More precisely, we have:

Lemma 2.2. *There exists $C_2 > 0$ (depending only on α, V) such that, if $M(t) \geq C_2$ for some $t \in [0,T]$ and $\mathcal{E}(\xi,t) \geq M(t) - 1$ for some $\xi \in \mathbb{R}$, then $\partial_t \mathcal{E}(\xi,t) \leq -1$.*

Assuming this result to be true, we now conclude the proof of Proposition 2.1. It follows readily from Lemma 2.2 that $M(t) \leq \max(C_2, M(0) - t)$ for all $t \in [0,T]$, an estimate which holds for any solution $w \in \mathcal{C}^0([0,T], X)$ of (2.5). This shows that any solution of (2.5) stays bounded in X for positive times (hence can be extended to a global solution), and that (2.4) holds with $K_\infty = C_1 C_2$. □

Proof of Lemma 2.2. We use the same notations as in the proof of Proposition 2.1. Let $C_3 = 2(1 + 2b/\kappa)$, and take $L > 0$ large enough so that $e^{\kappa L} \geq 3$. Fix also $\xi \in \mathbb{R}$ and $t \in [0,T]$. If $\mathcal{F}(\xi,t) \geq C_3$, then $\partial_t \mathcal{E}(\xi,t) \leq -1$ by (2.10). On the other hand, if $\mathcal{F}(\xi,t) \leq C_3$, there exists $C_4 > 0$ (depending on α, V, L, and C_3) such that

$$\int_{\xi-L}^{\xi+L} (T_\xi \rho)(x)\, e(u,u_x,u_t)(x,t)\,\mathrm{d}x \;\leq\; C_4 \;, \tag{2.11}$$

where $e(u,u_x,u_t) = \alpha^2 u_t^2 + \alpha u_x^2 + 2\alpha \overline{V}(u) + \frac{1}{2}u^2 + \alpha u u_t \geq 0$. Inequality (2.11) holds because $\mathcal{F}(\xi,t)$ controls the norm of (u,u_t) in $H^1([\xi-L,\xi+L]) \times L^2([\xi-L,\xi+L])$. As a consequence of (2.7), (2.11) at least one of the following inequalities holds:

$$\begin{aligned} \text{either} \quad & \int_{\xi+L}^{\infty} (T_\xi \rho)(x)\, e(u,u_x,u_t)(x,t)\,\mathrm{d}x \geq \frac{1}{2}(\mathcal{E}(\xi,t) - C_4) \;, \\ \text{or} \quad & \int_{-\infty}^{\xi-L} (T_\xi \rho)(x)\, e(u,u_x,u_t)(x,t)\,\mathrm{d}x \geq \frac{1}{2}(\mathcal{E}(\xi,t) - C_4) \;. \end{aligned} \tag{2.12}$$

Suppose for instance that the first inequality in (2.12) holds. Then

$$\begin{aligned} \mathcal{E}(\xi+L,t) \;&\geq\; \int_{\xi+L}^{\infty} (T_{\xi+L}\rho)(x)\, e(u,u_x,u_t)(x,t)\,\mathrm{d}x \\ &\geq\; 3\int_{\xi+L}^{\infty} (T_\xi \rho)(x)\, e(u,u_x,u_t)(x,t)\,\mathrm{d}x \;\geq\; \frac{3}{2}(\mathcal{E}(\xi,t) - C_4) \;, \end{aligned}$$

because $(T_{\xi+L}\rho)(x) \geq 3(T_\xi \rho)(x)$ for all $x \geq \xi + L$, by assumption on L. Using a similar argument in the other case we conclude that, if $\mathcal{F}(\xi,t) \leq C_3$, then

$$\max\Big(\mathcal{E}(\xi+L,t), \mathcal{E}(\xi-L,t)\Big) \;\geq\; \frac{3}{2}(\mathcal{E}(\xi,t) - C_4) \;. \tag{2.13}$$

Now, fix $C_5 > 3(C_4 + 1)$. If $M(t) \geq C_5$ and $\mathcal{E}(\xi,t) \geq M(t) - 1$, we claim that $\mathcal{F}(\xi,t) > C_3$, so that $\partial_t \mathcal{E}(\xi,t) \leq -1$ by (2.10). Indeed, if $\mathcal{F}(\xi,t) \leq C_3$, it follows from (2.13) that

$$M(t) \;\geq\; \max\Big(\mathcal{E}(\xi+L,t), \mathcal{E}(\xi-L,t)\Big) \;\geq\; \frac{3}{2}(M(t) - 1 - C_4) \;,$$

which contradicts the assumption that $M(t) \geq C_5$. □

Remark: The proof of Proposition 2.1 can be simplified if we assume, in addition to (1.3), that $uV'(u) \ge a'V(u) - b'$ for some positive constants a', b', but Lemma 2.2 allows us to avoid this unnecessary assumption.

To conclude this section, we show that the solutions of (1.1) given by Proposition 2.1 are locally asymptotically compact, in the following sense:

Proposition 2.3. *Let $u \in \mathcal{C}^0([0,+\infty), H^1_{\mathrm{ul}}(\mathbb{R})) \cap \mathcal{C}^1([0,+\infty), L^2_{\mathrm{ul}}(\mathbb{R}))$ be a solution of (1.1), and let $\{(x_n,t_n)\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}\times\mathbb{R}_+$ such that $t_n \to +\infty$ as $n \to \infty$. Then there exists a subsequence, still denoted (x_n,t_n), and a solution $\bar u \in \mathcal{C}^0(\mathbb{R}, H^1_{\mathrm{ul}}(\mathbb{R})) \cap \mathcal{C}^1(\mathbb{R}, L^2_{\mathrm{ul}}(\mathbb{R}))$ of (1.1) such that, for all $L > 0$ and all $T > 0$,*

$$\sup_{t\in[-T,T]} \Bigl(\|u(x_n+\cdot,t_n+t)-\bar u(\cdot,t)\|_{H^1([-L,L])}+\|u_t(x_n+\cdot,t_n+t)-\bar u_t(\cdot,t)\|_{L^2([-L,L])}\Bigr) \xrightarrow[n\to\infty]{} 0\ .$$

In other words, after extracting a subsequence, we can assume that the sequence $\{u(x_n+x,t_n+t)\}$ converges in $\mathcal{C}^0([-T,T], H^1_{\mathrm{loc}}(\mathbb{R})) \cap \mathcal{C}^1([-T,T], L^2_{\mathrm{loc}}(\mathbb{R}))$ towards a solution $\bar u(x,t)$ of (1.1), for any $T > 0$.

Proof: As in the proof of Proposition 2.1, we set $w = (u, u_t)$ and we consider Eq. (2.5) instead of Eq. (1.1). If $w_0 \in X$, the solution of (2.5) with initial data w_0 has the following representation:

$$w(t) \;=\; e^{At}w_0 + \int_0^t e^{A(t-s)}F(w(s))\,\mathrm{d}s \;\equiv\; w_1(t) + w_2(t)\ .$$

As is easily verified, there exists $C_6 > 0$ and $\mu > 0$ such that $\|e^{At}\|_{\mathcal{L}(X)} \le C_6\, e^{-\mu t}$ for all $t \ge 0$ (this estimate will be established in a more general setting in Section 9, Lemma 9.2). Thus $w_1(t) = e^{At}w_0$ converges exponentially to zero as $t \to +\infty$, and can therefore be neglected. On the other hand, by Proposition 2.1, there exists $C_7 > 0$ such that $\|w(t)\|_X \le C_7$ for all $t \ge 0$. As F maps X into $Y = D(A)$ and is Lipschitz on bounded sets, there exists $C_8 > 0$ such that $\|AF(w)\|_X \le C_8$ whenever $\|w\|_X \le C_7$. Since $Aw_2(t) = \int_0^t e^{A(t-s)}AF(w(s))\,\mathrm{d}s$, we deduce that

$$\|Aw_2(t)\|_X \;\le\; C_6 \int_0^t e^{-\mu(t-s)}\|AF(w(s))\|_X\,\mathrm{d}s \;\le\; \frac{C_6 C_8}{\mu}\ , \quad t \ge 0\ ,$$

hence there exists $C_9 > 0$ such that $\|w_2(t)\|_Y \le C_9$ for all $t \ge 0$. In particular, given any $T > 0$, the sequence $\{w_2(x_n+\cdot,t_n-T)\}$ is bounded in $H^2([-L,L]) \times H^1([-L,L])$ for any $L > 0$. Extracting a subsequence and using a diagonal argument, we can assume that there exists $\bar w_0 \in H^2_{\mathrm{loc}}(\mathbb{R}) \times H^1_{\mathrm{loc}}(\mathbb{R})$ such that, for any $L > 0$,

$$w_2(x_n+\cdot,t_n-T) \xrightarrow[n\to\infty]{} \bar w_0 \quad \text{in} \quad H^1([-L,L]) \times L^2([-L,L])\ . \tag{2.14}$$

By construction $\|\bar w_0\|_Y \le C_9$, hence in particular $\bar w_0 \in X$. Note that (2.14) still holds if we replace w_2 by the full solution w, because $\|w_1(\cdot,t)\|_X$ converges to zero. Finally, let $\bar w(t) \in \mathcal{C}^0([-T,+\infty), X)$ be the solution of (2.5) with initial data $w(\cdot,-T) = \bar w_0$. Since the evolution defined by (2.5) has a finite speed of propagation, it is clear that the solution

$\bar{w}(t)$ depends continuously on the initial data $\bar{w}_0$ in the topology of $H^1_{\rm loc}(\mathbb{R}) \times L^2_{\rm loc}(\mathbb{R})$, uniformly in time on compact intervals. Thus it follows from (2.14) that, for all $L > 0$,

$$\sup_{t\in[-T,T]} \|w(x_n + \cdot, t_n + t) - \bar{w}(\cdot, t)\|_{H^1([-L,L])\times L^2([-L,L])} \xrightarrow[n\to\infty]{} 0 \ .$$

Repeating the argument for larger T and using another diagonal extraction, we conclude the proof of Proposition 2.3. □

3 Pinching at infinity and splitting of the solution

In this section we prove that, if the initial data satisfy the boundary conditions (1.9),(1.10), the solution $u(x,t)$ of Eq. (1.1) has the same properties for all positive times. As a consequence, we show that any such solution can be decomposed into a principal part $v(x,t)$ which is compactly supported to the right, and a small remainder $r(x,t)$ which decays exponentially to zero as $t \to +\infty$.

We first verify that, due to assumptions (1.4), (1.5), the homogeneous equilibria $u = 0$ and $u = 1$ are stable steady states of Eq. (1.1). Let $(u_0, u_1) \in X = H^1_{\rm ul}(\mathbb{R}) \times L^2_{\rm ul}(\mathbb{R})$, and let (u, u_t) be the solution of Eq. (1.1) with initial data (u_0, u_1) given by Proposition 2.1.

Lemma 3.1. *There exist positive constants K_i, δ_i, μ_i for $i = 0, 1$ such that*
a) *If $\|(u_0, u_1)\|_X^2 \le \delta_0$, then $\|(u(\cdot,t), u_t(\cdot,t))\|_X^2 \le K_0\|(u_0, u_1)\|_X^2\, e^{-\mu_0 t}$ for all $t \ge 0$.*
b) *If $\|(u_0-1, u_1)\|_X^2 \le \delta_1$, then $\|(u(\cdot,t)-1, u_t(\cdot,t))\|_X^2 \le K_1\|(u_0-1, u_1)\|_X^2\, e^{-\mu_1 t}$ for all $t \ge 0$.*

Proof: It is sufficient to prove **a)**, the other case being similar. Let $\beta_0 = V''(0) > 0$, and choose $\varepsilon_0 > 0$ small enough so that

$$\frac{\beta_0}{2} \le V''(u) \le 2\beta_0 \ , \quad \text{for all } u \in [-\varepsilon_0, \varepsilon_0] \ . \tag{3.1}$$

In particular, we have

$$\frac{\beta_0 u^2}{2} \le uV'(u) \le 2\beta_0 u^2 \ , \quad \text{and} \quad \frac{\beta_0 u^2}{4} \le V(u) \le \beta_0 u^2 \ , \tag{3.2}$$

whenever $|u| \le \varepsilon_0$. In analogy with (2.7), we introduce the functional

$$\mathcal{E}_0(\xi, t) = \int_{\mathbb{R}} (T_\xi\rho)(x)\Big(\alpha^2 u_t^2 + \alpha u_x^2 + 2\alpha V(u) + \frac{1}{2}u^2 + \alpha u u_t\Big)(x,t)\,{\rm d}x \ ,$$

where $\rho(x) = \exp(-\kappa|x|)$ and $\kappa > 0$ is small enough so that $2\sqrt{\alpha}\kappa \le 1$ and $2\kappa^2 \le \beta_0$. If $\|u(\cdot,t)\|_{L^\infty}^2 \le 2\|u(\cdot,t)\|_{H^1_{\rm ul}}^2 \le \varepsilon_0^2$, it follows from (3.2) and from the definitions (2.1)–(2.3) that

$$C_0^{-1}\|(u(\cdot,t), u_t(\cdot,t))\|_X^2 \le \sup_{\xi\in\mathbb{R}} \mathcal{E}_0(\xi, t) \le C_0\|(u(\cdot,t), u_t(\cdot,t))\|_X^2 \ ,$$

for some $C_0 > 1$. Under the same assumption, we find as in the proof of Proposition 2.1:

$$\begin{aligned}\partial_t \mathcal{E}_0(\xi,t) &= -\int_{\mathbb{R}} (T_\xi \rho)(\alpha u_t^2 + u_x^2 + uV'(u))\,\mathrm{d}x - \int_{\mathbb{R}} (T_\xi\rho)'(uu_x + 2\alpha u_x u_t)\,\mathrm{d}x \\ &\le -\int_{\mathbb{R}} (T_\xi\rho)\Big(\frac{\alpha}{2}u_t^2 + \frac{1}{2}u_x^2 + \frac{\beta_0}{4}u^2\Big)\,\mathrm{d}x \le -\mu_0 \mathcal{E}_0(\xi,t)\ ,\end{aligned}$$

for some $\mu_0 > 0$. Now, let $K_0 = C_0^2$ and choose $\delta_0 > 0$ small enough so that $2K_0\delta_0 < \varepsilon_0^2$. If $\|(u_0,u_1)\|_X^2 \le \delta_0$, the inequalities above imply that the solution (u,u_t) satisfies $\|(u(\cdot,t),u_t(\cdot,t))\|_X^2 \le K_0\|(u_0,u_1)\|_X^2\, e^{-\mu_0 t}$ for all $t \ge 0$. In particular, $\|u(\cdot,t)\|_{L^\infty}^2 \le 2\|u(\cdot,t)\|_{H^1_{\mathrm{ul}}}^2 \le \varepsilon_0^2\, e^{-\mu_0 t}$ for all $t \ge 0$. ☐

From now on, we assume that the initial data $(u_0,u_1) \in X$ satisfy the assumptions (1.9), (1.10) for some $\delta \le \min(\delta_0,\delta_1)/2$, and we let $u \in \mathcal{C}^0([0,+\infty), H^1_{\mathrm{ul}}(\mathbb{R})) \cap \mathcal{C}^1([0,+\infty), L^2_{\mathrm{ul}}(\mathbb{R}))$ be the solution of (1.1) given by Theorem 1.1. Using Lemma 3.1 and the finite speed of propagation we show that, for all $t \ge 0$, the solution $u(x,t)$ stays close for large $|x|$ to the homogenous equilibria $u = 0$ and $u = 1$.

Proposition 3.2. *If $\delta \le \min(\delta_0,\delta_1)/2$, the solution of* (1.1) *given by Theorem 1.1 satisfies, for all $t \ge 0$,*

$$\limsup_{\xi\to+\infty} \int_\xi^{\xi+1} \Big(u(x,t)^2 + u_x(x,t)^2 + u_t(x,t)^2\Big)\,\mathrm{d}x \le K_0\delta_0\, e^{-\mu_0 t}\ , \qquad (3.3)$$

$$\limsup_{\xi\to-\infty} \int_\xi^{\xi+1} \Big((u(x,t)-1)^2 + u_x(x,t)^2 + u_t(x,t)^2\Big)\,\mathrm{d}x \le K_1\delta_1\, e^{-\mu_1 t}. \qquad (3.4)$$

Proof: We only prove the first inequality, the second one being similar. Take $\xi_0 \in \mathbb{R}$ such that

$$\int_\xi^{\xi+1} \Big(u_0(x)^2 + u_0'(x)^2 + u_1(x)^2\Big)\,\mathrm{d}x \le \frac{3\delta_0}{4}\ , \quad \text{for all } \xi \ge \xi_0 - 4\ .$$

We consider the modified initial data $(r_0,r_1) \in X$ defined by

$$r_0(x) = \theta(x-\xi_0)u_0(x)\ , \quad r_1(x) = \theta(x-\xi_0)u_1(x)\ , \quad x \in \mathbb{R}\ , \qquad (3.5)$$

where $\theta(x) = \min(1,(1+x/4)_+)$ satisfies $\theta(x) = 1$ for $x \ge 0$, $\theta(x) = 0$ for $x \le -4$, and $|\theta'(x)| \le 1/4$ for all x. By construction $(r_0(x),r_1(x)) = (u_0(x),u_1(x))$ for all $x \ge \xi_0$, and

$$\|(r_0,r_1)\|_X^2 \le \frac{4}{3}\sup_{\xi\ge\xi_0-4}\int_\xi^{\xi+1}\Big(u_0(x)^2 + u_0'(x)^2 + u_1(x)^2\Big)\,\mathrm{d}x \le \delta_0\ .$$

If $(r,r_t) \in \mathcal{C}^0([0,+\infty),X)$ is the solution of (1.1) with initial data (r_0,r_1), we know from Lemma 3.1 that

$$\|(r(\cdot,t),r_t(\cdot,t))\|_X^2 \le K_0\delta_0\, e^{-\mu_0 t}\ , \quad \text{for all } t \ge 0\ . \qquad (3.6)$$

On the other hand, the finite speed of propagation implies that $u(x,t) = r(x,t)$ for all $t \ge 0$ and all $x \ge \xi_0 + t/\sqrt{\alpha}$. Both observations together imply (3.3). ☐

Decomposition of the solution: The proof of Proposition 3.2 provides us with a useful decomposition of the solution of (1.1). Let

$$u(x,t) \;=\; v(x,t) + r(x,t) \;, \quad x \in \mathbb{R} \;, \quad t \ge 0 \;, \tag{3.7}$$

where $r(x,t)$ is the solution of (1.1) associated to the initial data (r_0, r_1) defined in (3.5). By construction, the principal part $v(x,t)$ vanishes identically for $x \ge \xi_0 + t/\sqrt{\alpha}$, and satisfies the modified equation

$$\alpha v_{tt} + v_t \;=\; v_{xx} - V'(v+r) + V'(r) \;, \tag{3.8}$$

supplemented with the initial data $(v_0, v_1) = (u_0 - r_0, u_1 - r_1)$. If we define

$$f(v,r) \;=\; V'(v) + V'(r) - V'(v+r) \;=\; -vr \int_0^1\!\!\int_0^1 V'''(tv+sr)\,\mathrm{d}t\,\mathrm{d}s \;, \tag{3.9}$$

we can rewrite (3.8) in the form

$$\alpha v_{tt} + v_t \;=\; v_{xx} - V'(v) + f(v,r) \;. \tag{3.10}$$

The main advantage of working with (3.10) instead of (1.1) is that the energy functional (1.14) (with u replaced by v) is well-defined for all $c > 0$ since $v(x,t)$ is compactly supported to the right. The price to pay is the additional term $f(v,r)$ in (3.10), which we shall treat as a perturbation. Remark that, since $v(x,t)$ and $r(x,t)$ stay uniformly bounded for all $t \ge 0$, the formula (3.9) shows that there exists $K_2 > 0$ such that

$$|f(v(x,t), r(x,t))| \;\le\; K_2\,|v(x,t)|\,|r(x,t)| \;, \quad x \in \mathbb{R} \;, \quad t \ge 0 \;. \tag{3.11}$$

Moreover, we know that $\|r(\cdot,t)\|_{L^\infty}^2 \le 2\|r(\cdot,t)\|_{H^1_{\mathrm{ul}}}^2 \le \varepsilon_0^2\, e^{-\mu_0 t}$ for all $t \ge 0$, hence (3.10) is really a small perturbation of (1.1) for large times. In particular, the asymptotic compactness property stated in Proposition 2.3 holds for the solution $v(x,t)$ of (3.10), and by Proposition 2.1 there exists $M_0 > 0$ such that

$$\|v(\cdot,t)\|_{H^1_{\mathrm{ul}}}^2 + \|v_t(\cdot,t)\|_{L^2_{\mathrm{ul}}}^2 \;\le\; M_0^2 \;, \quad \text{for all } t \ge 0 \;. \tag{3.12}$$

The invasion point: As is explained in [15, 38], to control the behavior of the solution $u(x,t)$ of (1.1) using the energy functionals (1.14) it is necessary to track for all times the approximate position of the front interface. Since $r(x,t)$ converges uniformly to zero as $t \to +\infty$, this can be done for the solution $v(x,t)$ of (3.10) instead of $u(x,t)$. We thus introduce the invasion point $\overline{x}(t) \in \mathbb{R}$ defined for any $t \ge 0$ by

$$\overline{x}(t) \;=\; \sup\{x \in \mathbb{R} \,|\, |v(x,t)| \ge \varepsilon_0\} \;, \tag{3.13}$$

where ε_0 is as in (3.1). It is clear that $\overline{x}(t) < \xi_0 + t/\sqrt{\alpha}$ since $v(x,t)$ vanishes identically for larger values of x. In the same way, using (3.4), one can prove that there exists $\xi_1 \in \mathbb{R}$ such that $\overline{x}(t) > \xi_1 - t/\sqrt{\alpha}$ for all $t \ge 0$. Note that $\overline{x}(t)$ is not necessarily a continuous function of t, although it follows from the definition (3.13) that $\overline{x}(t)$ is upper semi-continuous.

4 Energy estimates in a Galilean frame

As is explained in the introduction, the proof of Theorem 1.1 is based on the existence of Lyapunov functions for Eq. (1.1) in uniformly translating frames. The aim of this section is to define these functions rigorously and to study their basic properties.

Let $u(x,t)$ be a solution of (1.1) whose initial data satisfy the assumptions of Theorem 1.1. Given any $c > 0$, we go to a uniformly translating frame by setting, as in (1.12), $u(x,t) = u_c(\sqrt{1+\alpha c^2}\,x - ct\,,\,t)$. To avoid confusions, we always denote by $y = \sqrt{1+\alpha c^2}\,x - ct$ the space variable in the moving frame. Note that the *physical speed* $s \in (0, 1/\sqrt{\alpha})$ of the frame is related to the *parabolic speed* $c > 0$ by the formulas

$$s \;=\; \frac{c}{\sqrt{1+\alpha c^2}}\;, \qquad c \;=\; \frac{s}{\sqrt{1-\alpha s^2}}\;. \tag{4.1}$$

If $u(x,t)$ is decomposed according to (3.7), then $u_c(y,t) = v_c(y,t) + r_c(y,t)$ where

$$v_c(y,t) \;=\; v\Bigl(\frac{y+ct}{\sqrt{1+\alpha c^2}}\,,\,t\Bigr)\;, \quad \text{and} \quad r_c(y,t) \;=\; r\Bigl(\frac{y+ct}{\sqrt{1+\alpha c^2}}\,,\,t\Bigr)\;. \tag{4.2}$$

By construction, both v_c and r_c belong to $\mathcal{C}^0([0,+\infty), H^1_{\rm ul}(\mathbb{R})) \cap \mathcal{C}^1([0,+\infty), L^2_{\rm ul}(\mathbb{R}))$. Moreover, from (3.12) and Lemma 3.1, we know that

$$\|v_c(t)\|_{L^\infty}^2 \;\le\; 2\|v(t)\|_{H^1_{\rm ul}}^2 \;\le\; M_0^2\;, \quad \|r_c(t)\|_{L^\infty}^2 \;\le\; 2\|r(t)\|_{H^1_{\rm ul}}^2 \;\le\; \varepsilon_0^2\, e^{-\mu_0 t}\;, \tag{4.3}$$

for all $t \ge 0$. In view of (1.13), (3.10), the evolution equations satisfied by v_c, r_c read

$$\left\{\begin{array}{rcl} \alpha\ddot r_c + \dot r_c - 2\alpha c\dot r_c' &=& r_c'' + cr_c' - V'(r_c)\;, \\ \alpha\ddot v_c + \dot v_c - 2\alpha c\dot v_c' &=& v_c'' + cv_c' - V'(v_c) + f(v_c,r_c)\;, \end{array}\right. \tag{4.4}$$

where $f(v_c,r_c) = -V'(v_c+r_c) + V'(v_c) + V'(r_c)$. Here and in the rest of the text, to simplify the notation and to avoid double subscripts, we denote $\dot v_c(y,t) = \partial_t v_c(y,t)$, $v_c'(y,t) = \partial_y v_c(y,t)$, and similarly for r_c. In analogy with (3.13), we also define the *invasion point* in the moving frame by

$$\overline{y}_c(t) \;=\; \sqrt{1+\alpha c^2}\,\overline{x}(t) - ct \;=\; \sup\{y \in \mathbb{R} \,|\, |v_c(y,t)| \ge \varepsilon_0\}\;. \tag{4.5}$$

4.1 The energy functional

In a moving frame with parabolic speed $c > 0$, the energy functional involves an exponentially growing weight e^{cy}, see (1.14). It is thus natural to introduce the following weighted spaces:

$$\begin{aligned} L^2_c(\mathbb{R}) \;&=\; \{u \in L^2_{\rm loc}(\mathbb{R}) \,|\, e^{cy/2}u \in L^2(\mathbb{R})\}\;, \\ H^1_c(\mathbb{R}) \;&=\; \{u \in H^1_{\rm loc}(\mathbb{R}) \,|\, e^{cy/2}u \in L^2(\mathbb{R}) \text{ and } e^{cy/2}u' \in L^2(\mathbb{R})\}\;. \end{aligned} \tag{4.6}$$

Since $v_c(\cdot,t) \in H^1_{\rm ul}(\mathbb{R})$ and $v_c(y,t)$ vanishes for all sufficiently large $y > 0$, it is clear that $v_c(\cdot,t)$ belongs to $H^1_c(\mathbb{R})$ for any $c > 0$. Similarly, $\dot v_c(\cdot,t)$ belongs to $L^2_c(\mathbb{R})$. The following quantity is thus well-defined for any $y_0 \in \mathbb{R}$ and all $t \ge 0$:

$$E_c(y_0,t) \;=\; \int_{\mathbb{R}} e^{cy}\Bigl(\frac{\alpha}{2}|\dot v_c|^2 + \frac{1}{2}|v_c'|^2 + V(v_c)\Bigr)(y_0+y,t)\,{\rm d}y\;. \tag{4.7}$$

We shall refer to $E_c(y_0,t)$ as the *energy* of the solution $v_c(y,t)$ in the moving frame. The translation parameter y_0 is introduced here for later convenience. Changing y_0 results in a simple rescaling, as is clear from the identity

$$E_c(y_0,t) \;=\; e^{c(y_1-y_0)} E_c(y_1,t)\;. \tag{4.8}$$

Due to the term $f(v_c,r_c)$ in (4.4), the energy $E_c(y_0,t)$ is not necessarily a decreasing function of time. Indeed, a formal calculation gives

$$\partial_t E_c(y_0,t) \;=\; -(1+\alpha c^2)\int_{\mathbb{R}} e^{cy}|\dot v_c(y_0+y,t)|^2\,\mathrm{d}y + R_c(y_0,t)\;, \tag{4.9}$$

where

$$R_c(y_0,t) \;=\; \int_{\mathbb{R}} e^{cy}(f(v_c,r_c)\dot v_c)(y_0+y,t)\,\mathrm{d}y\;. \tag{4.10}$$

Using (3.11) and (4.3), it is easy to verify that $R_c(y_0,t)$ is well-defined for all $t\ge 0$ and depends continuously on time. A classical argument then shows that $E_c(y_0,t)$ is indeed differentiable with respect to t and that (4.9) holds for all $t\ge 0$. The purpose of this section is to show that, in appropriate situations, the remainder term R_c in (4.9) is a negligible quantity which does not really affect the decay of the energy. Our first result in this direction is:

Lemma 4.1. *There exists a positive constant K_3, independent of c, such that*

$$|R_c(y_0,t)| \;\le\; K_3\,e^{-\mu t}\Big(E_c(y_0,t) + \frac{K_3}{c}\,e^{c(\overline{y}_c(t)-y_0)}\Big)\;, \tag{4.11}$$

for all $y_0\in\mathbb{R}$ and all $t\ge 0$, where $\mu=\mu_0/2$ with $\mu_0>0$ as in Lemma 3.1.

Proof: Using (3.11), (4.3), and (4.10), we obtain

$$\begin{aligned}|R_c(y_0,t)| \;&\le\; K_2\varepsilon_0\,e^{-\mu t}\int_{\mathbb{R}} e^{c(y-y_0)}|\dot v_c\,v_c|(y,t)\,\mathrm{d}y\\ &\le\; \frac{K_2\varepsilon_0}{2}\,e^{-\mu t}\int_{\mathbb{R}} e^{c(y-y_0)}(|\dot v_c|^2+|v_c|^2)(y,t)\,\mathrm{d}y\;.\end{aligned}$$

If $y\ge\overline{y}_c(t)$, then $|v_c(y,t)|\le\varepsilon_0$ by (4.5), hence $|v_c(y,t)|^2\le(4/\beta_0)V(v_c(y,t))$ by (3.2). Thus

$$\frac{1}{2}(|\dot v_c(y,t)|^2+|v_c(y,t)|^2) \;\le\; C\Big(\frac{\alpha}{2}|\dot v_c(y,t)|^2+\frac{1}{2}|v_c'(y,t)|^2+V(v_c(y,t))\Big)\;,$$

where $C=\max(\alpha^{-1},2\beta_0^{-1})$. If $y\le\overline{y}_c(t)$, we can bound

$$\begin{aligned}\frac{1}{2}(|\dot v_c(y,t)|^2+|v_c(y,t)|^2) \;&\le C\Big(\frac{\alpha}{2}|\dot v_c(y,t)|^2+\frac{1}{2}|v_c'(y,t)|^2+V(v_c(y,t))\Big)\\ &\quad+\frac{1}{2}\|v_c(t)\|_{L^\infty}^2 + C|\min V|\;.\end{aligned}$$

Combining these estimates and using (4.3), we thus obtain

$$
\begin{aligned}
|R_c(y_0,t)| \;&\le\; K_2\varepsilon_0\, e^{-\mu t}\left(CE_c(y_0,t) + \Big(\frac{M_0^2}{2} + C|\min V|\Big)\int_{-\infty}^{\overline{y}_c(t)} e^{c(y-y_0)}\,\mathrm{d}y\right) \\
&\le\; K_3\, e^{-\mu t}\left(E_c(y_0,t) + \frac{K_3}{c}\, e^{c(\overline{y}_c(t)-y_0)}\right)\;,
\end{aligned}
$$

which is the desired result. □

The following corollary of Lemma 4.1 will turn out to be useful:

Proposition 4.2. *Assume that the invasion point satisfies, for some* $c_+ > 0$,

$$
\limsup_{t\to+\infty} \frac{\overline{y}_{c_+}(t)}{t} \;\le\; 0\;.
$$

Then, there exist $\eta > 0$ *and* $K_4 > 0$ *such that, for all* $c \in [c_+ - \eta, c_+ + \eta]$*, all* $y_0 \in \mathbb{R}$*, and all* $t_1 \ge t_0 \ge 0$*, one has*

$$
E_c(y_0,t_1) \;\le\; K_4 \max(E_c(y_0,t_0)\,,\, e^{-cy_0})\;. \tag{4.12}
$$

Moreover

$$
|R_c(y_0,t)| \;\le\; K_4\, e^{-\mu t/2} \max(E_c(y_0,t_0)\,,\, e^{-cy_0})\;, \quad \textit{for all } t \ge t_0\;. \tag{4.13}
$$

Proof: By assumption, for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that $\overline{y}_{c_+}(t) \le \varepsilon t + C_\varepsilon$ for all $t \ge 0$. In a frame moving at parabolic speed c, this bound becomes

$$
\overline{y}_c(t) \;\le\; \left(\sqrt{\frac{1+\alpha c^2}{1+\alpha c_+^2}}(c_+ + \varepsilon) - c\right) t \;+\; \sqrt{\frac{1+\alpha c^2}{1+\alpha c_+^2}}\, C_\varepsilon\;.
$$

Thus, if we choose $\varepsilon > 0$ small enough, there exist $\eta \in (0, c_+)$ and $C_1 > 0$ such that, for all $c \in [c_+ - \eta, c_+ + \eta]$ and all $t \ge 0$, we have $c\overline{y}_c(t) \le (\mu t)/2 + C_1$. Using (4.9) and Lemma 4.1, we deduce that

$$
\partial_t E_c(y_0,t) \;\le\; |R_c(y_0,t)| \;\le\; K_3\, e^{-\mu t}(E_c(y_0,t) + C_2\, e^{\mu t/2 - cy_0})\;, \tag{4.14}
$$

for some $C_2 > 0$. Integrating this differential inequality between t_0 and t_1, we obtain

$$
\begin{aligned}
E_c(y_0,t_1) \;&\le\; e^{\frac{K_3}{\mu}(e^{-\mu t_0} - e^{-\mu t_1})} E_c(y_0,t_0) + K_3 C_2 \int_{t_0}^{t_1} e^{\frac{K_3}{\mu}(e^{-\mu t} - e^{-\mu t_1})}\, e^{-\mu t/2 - cy_0}\,\mathrm{d}t \\
&\le\; K_4 \max(E_c(y_0,t_0)\,,\, e^{-cy_0})\;,
\end{aligned}
$$

which proves (4.12). Estimate (4.13) is a direct consequence of (4.12) and (4.14). □

Remark: Of course, if the initial data u_0, u_1 decay rapidly enough as $x \to +\infty$, the decomposition (3.7) is not needed and we can use the energy (1.14) instead of (4.7). This is the point of view adopted in [15]. In a first reading of the paper, it is therefore possible to set $r = 0$ everywhere, in which case the remainder term $R_c(y_0,t)$ disappears from (4.9) and the energy $E_c(y_0,t)$ is a true Lyapunov function. However, once the invasion point is under control, the results of this section show that $R_c(y_0,t)$ becomes really negligible for large times. Thus the general case can be seen as a perturbation of the particular situation where $r = 0$, and the structure of the proof is the same in both cases.

4.2 A Poincaré inequality

As was already observed in [33] and [15], Poincaré inequalities hold in the weighted space $H^1_c(\mathbb{R})$ if $c > 0$.

Proposition 4.3. *Let $c > 0$ and let $v_c \in H^1_c(\mathbb{R})$. Then $e^{cy}|v_c(y)|^2 \to 0$ as $y \to +\infty$. Moreover, for any $y_1 \in \mathbb{R}$,*

$$\frac{c^2}{4}\int_{y_1}^{\infty} e^{cy}|v_c(y)|^2\,\mathrm{d}y \;\le\; \int_{y_1}^{\infty} e^{cy}|v_c'(y)|^2\,\mathrm{d}y\;, \tag{4.15}$$

and

$$ce^{cy_1}|v_c(y_1)|^2 \;\le\; \int_{y_1}^{\infty} e^{cy}|v_c'(y)|^2\,\mathrm{d}y\;. \tag{4.16}$$

Proof: A simple integration shows that, for all $y_1 \le y_2$,

$$e^{cy_2}|v_c(y_2)|^2 - e^{cy_1}|v_c(y_1)|^2 \;=\; 2\int_{y_1}^{y_2} e^{cy}v_c'(y)v_c(y)\,\mathrm{d}y + c\int_{y_1}^{y_2} e^{cy}|v_c(y)|^2\,\mathrm{d}y\;.$$

When y_2 goes to $+\infty$, both integrals in the right-hand side have a finite limit since $v_c \in H^1_c(\mathbb{R})$. Thus the first term in the left-hand side also has a limit, which is necessarily zero since $y \mapsto e^{cy}|v_c(y)|^2 \in L^1(\mathbb{R})$. It follows that

$$e^{cy_1}|v_c(y_1)|^2 \;\le\; 2\int_{y_1}^{\infty} e^{cy}|v_c'(y)v_c(y)|\,\mathrm{d}y - c\int_{y_1}^{\infty} e^{cy}|v_c(y)|^2\,\mathrm{d}y\;. \tag{4.17}$$

Now, for any $d > -c$, we have $|2v_cv_c'| \le (c+d)|v_c|^2 + \frac{1}{c+d}|v_c'|^2$. Inserting this bound in (4.17) we find

$$e^{cy_1}|v_c(y_1)|^2 \le \frac{1}{c+d}\int_{y_1}^{\infty} e^{cy}|v_c'(y)|^2\,\mathrm{d}y + d\int_{y_1}^{\infty} e^{cy}|v_c(y)|^2\,\mathrm{d}y\;,$$

from which (4.15) follows by taking $d = -c/2$ and (4.16) by choosing $d = 0$. □

The Poincaré inequality implies the following important lower bound on the energy. We recall that, for all $y \ge \overline{y}_c(t)$, one has $|v_c(y,t)| \le \varepsilon_0$ by (4.5), so that $V(v_c(y,t)) \ge 0$ by (3.2). Thus

$$\begin{aligned} E_c(y_0,t) \;&=\; e^{-cy_0}\int_{-\infty}^{\overline{y}_c(t)} e^{cy}\Big(\frac{\alpha}{2}|\dot v_c|^2 + \frac{1}{2}|v_c'|^2 + V(v_c)\Big)(y,t)\,\mathrm{d}y \\ &\quad + e^{-cy_0}\int_{\overline{y}_c(t)}^{\infty} e^{cy}\Big(\frac{\alpha}{2}|\dot v_c|^2 + \frac{1}{2}|v_c'|^2 + V(v_c)\Big)(y,t)\,\mathrm{d}y \\ &\ge\; e^{-cy_0}\int_{-\infty}^{\overline{y}_c(t)} e^{cy}(\min V)\,\mathrm{d}y + e^{-cy_0}\frac{1}{2}\int_{\overline{y}_c(t)}^{\infty} e^{cy}|v_c'(y,t)|^2\,\mathrm{d}y\;. \end{aligned}$$

Using now (4.16) and the fact that $|v_c(\overline{y}_c(t),t)| = \varepsilon_0$, we obtain

$$E_c(y_0,t) \;\ge\; e^{c(\overline{y}_c(t)-y_0)}\Big(\frac{\min V}{c} + \frac{c\varepsilon_0^2}{2}\Big)\;. \tag{4.18}$$

5 Existence of the invasion speed

The purpose of this section is to show that the invasion point $\overline{x}(t)$ defined in (3.13) has a positive average speed as $t \to +\infty$:

Proposition 5.1. *The limit* $s_\infty = \lim_{t\to+\infty} \frac{\overline{x}(t)}{t}$ *exists and lies in the interval* $(0, \frac{1}{\sqrt{\alpha}})$.

We call s_∞ the *invasion speed* because this is the speed at which the front interface described by the solution $u(x,t)$ "invades" the steady state $u = 0$. We prove that $s_\infty < \frac{1}{\sqrt{\alpha}}$, which means that the invasion process is always "subsonic". As a side remark, we mention that this property may not hold in the monostable case, that is, if we drop the assumption that the equilibrium $u = 0$ is stable. For instance, if $h(x) = (1+e^x)^{-1}$, one can check that $u(x,t) = h(x - st)$ is a solution of (1.1) provided that

$$-V'(u) \;=\; u(1-u)(s + \gamma(1-2u)) \;, \quad \text{where } \gamma \;=\; \alpha s^2 - 1 \;.$$

If we choose $s > 0$ large enough so that $\gamma > 0$, the front $h(x - st)$ is supersonic, but in that case we also have $V''(0) < 0$, hence $u = 0$ is an unstable equilibrium of (1.1).

Our proof of Proposition 5.1 follows closely the method introduced in [38] and simplified in [15]. It is divided into three lemmas.

Lemma 5.2. *One has* $\limsup_{t\to+\infty} \frac{\overline{x}(t)}{t} < \frac{1}{\sqrt{\alpha}}$.

Proof: Choose $c > 0$ large enough so that $\frac{\min V}{c} + \frac{c\varepsilon_0^2}{2} > 0$. By (4.18), there exists $C_1 > 0$ such that $E_c(0,t) \ge C_1\, e^{c\overline{y}_c(t)}$ for all $t \ge 0$. Inserting this bound into (4.11), we see that there exists $C_2 > 0$ such that

$$\partial_t E_c(0,t) \;\le\; |R_c(0,t)| \;\le\; C_2\, e^{-\mu t} E_c(0,t) \;, \quad \text{for all } t \ge 0 \;.$$

If we integrate this differential inequality as in the proof of Proposition 4.2, we find that $E_c(0,t) \le C_3 E_c(0,0)$ for all $t \ge 0$. Going back to the lower bound $E_c(0,t) \ge C_1\, e^{c\overline{y}_c(t)}$, we conclude that $\overline{y}_c(t)$ is bounded from above, hence

$$\limsup_{t\to+\infty} \frac{\overline{x}(t)}{t} \;=\; \frac{1}{\sqrt{1+\alpha c^2}}\Big(c + \limsup_{t\to+\infty} \frac{\overline{y}_c(t)}{t}\Big) \;\le\; \frac{c}{\sqrt{1+\alpha c^2}} \;<\; \frac{1}{\sqrt{\alpha}} \;,$$

which is the desired result. □

Lemma 5.3. *One has* $\limsup_{t\to+\infty} \frac{\overline{x}(t)}{t} > 0$.

Proof: We argue by contradiction and assume that $\limsup(\overline{y}_c(t)/t) < 0$ for all $c > 0$. Using (4.9), (4.10) together with the bound (3.11), we find

$$\begin{aligned}\partial_t E_c(0,t) \;&=\; -(1+\alpha c^2)\int_{\mathbb{R}} e^{cy}|\dot v_c|^2(y,t)\,\mathrm{d}y + \int_{\mathbb{R}} e^{cy}(f(v_c,r_c)\dot v_c)(y,t)\,\mathrm{d}y \\ &\le\; \frac{1}{4}\int_{\mathbb{R}} e^{cy}|f(v_c,r_c)|^2(y,t)\,\mathrm{d}y \;\le\; \frac{K_2^2}{4}\int_{\mathbb{R}} e^{cy}|v_c(y,t)|^2|r_c(y,t)|^2\,\mathrm{d}y \;.\end{aligned}$$

Our goal is to bound the right-hand side by a quantity which is integrable in time and independent of c if c is sufficiently small. To do that, we fix $c' = 2\sqrt{\mu}$, where $\mu > 0$ is as in Lemma 4.1, and we assume that $c \in (0, c']$. Denoting $\rho = \sqrt{1+\alpha c^2}/\sqrt{1+\alpha c'^2}$ and using the definitions (4.2), we obtain the identity

$$\begin{aligned}\int_{\mathbb{R}} e^{cy}|v_c|^2|r_c|^2(y,t)\,\mathrm{d}y &= \int_{\mathbb{R}} e^{cy}|v_{c'}|^2|r_{c'}|^2(\rho^{-1}(y+ct)-c't\,,\,t)\,\mathrm{d}y \\ &= \rho\, e^{c(\rho c'-c)t}\int_{\mathbb{R}} e^{c\rho z}|v_{c'}|^2|r_{c'}|^2(z,t)\,\mathrm{d}z\ .\end{aligned}$$

Since $\rho \le 1$ and $c(\rho c'-c) \le c(c'-c) \le c'^2/4 = \mu$ by construction, we have

$$\int_{\mathbb{R}} e^{cy}|v_c(y,t)|^2|r_c(y,t)|^2\,\mathrm{d}y \;\le\; e^{\mu t}\left(\int_0^\infty e^{c'y}|v_{c'}|^2|r_{c'}|^2(y,t)\,\mathrm{d}y + \int_{-\infty}^0 |v_{c'}|^2|r_{c'}|^2(y,t)\,\mathrm{d}y\right)\ .$$

Remark that the right-hand side is now independent of c. To bound the first integral, we proceed as in the proof of Lemma 4.1. Since $\overline{y}_{c'}(t)$ is bounded from above by assumption, so is $E_{c'}(0,t)$ by Proposition 4.2 and we obtain

$$\begin{aligned}\int_0^\infty e^{c'y}|v_{c'}|^2|r_{c'}|^2(y,t)\,\mathrm{d}y &\le \varepsilon_0^2\, e^{-2\mu t}\int_{\mathbb{R}} e^{c'y}|v_{c'}(y,t)|^2\,\mathrm{d}y \\ &\le \varepsilon_0^2\, e^{-2\mu t}\Big(CE_{c'}(0,t) + (M_0^2 + C|\min V|)\frac{1}{c'}\,e^{c'\overline{y}_{c'}(t)}\Big) \;\le\; C_4\, e^{-2\mu t}\ ,\end{aligned}$$

for some $C_4 > 0$. To estimate the second integral we observe that $r(x,t) = 0$ for $x \le \xi_0 - 4 - t/\sqrt{\alpha}$, because the initial data (r_0, r_1) satisfy (3.5). Thus there exists $C_5 > 0$ such that $r_{c'}(y,t) = 0$ whenever $y \le -C_5(1+t)$, hence

$$\int_{-\infty}^0 |v_{c'}|^2|r_{c'}|^2(y,t)\,\mathrm{d}y \;\le\; C_5(1+t)\|v_{c'}(t)\|_{L^\infty}^2\|r_{c'}(t)\|_{L^\infty}^2 \;\le\; C_5(1+t)M_0^2\varepsilon_0^2\, e^{-2\mu t}\ .$$

Summarizing, we have shown the existence of a constant $C_6 > 0$ such that $\partial_t E_c(0,t) \le C_6(1+t)\,e^{-\mu t}$ for all $t \ge 0$ and all $c \in (0,c']$. In particular, $\int_0^\infty \partial_t E_c(0,t)\,\mathrm{d}t \le C_7 = C_6(1+\mu)/\mu^2$.

Now, if the initial data (u_0, u_1), or equivalently (v_0, v_1), satisfy the boundary condition (1.9) for some sufficiently small $\delta > 0$, it is straightforward to verify that

$$\frac{E_c(0,0)}{\sqrt{1+\alpha c^2}} = \int_{\mathbb{R}} e^{c\sqrt{1+\alpha c^2}\,x}\Big(\frac{\alpha}{2}|v_1 + s v_0'|^2 + \frac{1}{2(1+\alpha c^2)}|v_0'|^2 + V(v_0)\Big)(x)\,\mathrm{d}x \;\xrightarrow[c\to 0]{}\; -\infty\ , \quad (5.1)$$

because $V(1) < 0$. In particular we can take $c \in (0, c']$ small enough so that $E_c(0,0) \le -2C_7$. Then $E_c(0,t) \le -C_7$ for all $t \ge 0$, and since $E_c(0,t) \ge -e^{c\overline{y}_c(t)}|\min V|/c$ by (4.18), we conclude that $\overline{y}_c(t)$ is bounded from below. Thus $\limsup(\overline{y}_c(t)/t) \ge 0$, which is the desired contradiction. □

Lemma 5.4. *One has* $\displaystyle\liminf_{t\to+\infty}\frac{\overline{x}(t)}{t} = \limsup_{t\to+\infty}\frac{\overline{x}(t)}{t}\ .$

Proof: Again, we argue by contradiction and assume that

$$s_- \;\equiv\; \liminf_{t\to+\infty} \frac{\overline{x}(t)}{t} \;<\; \limsup_{t\to+\infty} \frac{\overline{x}(t)}{t} \;\equiv\; s_+ \;.$$

Then there exist two increasing sequences of times t_n and t'_n, both converging to $+\infty$, such that

$$\frac{\overline{x}(t_n)}{t_n} \;\xrightarrow[n\to\infty]{}\; s_+ \;, \quad \text{and} \quad \frac{\overline{x}(t'_n)}{t'_n} \;\xrightarrow[n\to\infty]{}\; s_- \;.$$

Given $T > 0$, we can assume in view of Proposition 2.3 that the sequence of functions $(v, \dot v)(\overline{x}(t_n) + \cdot, t_n + \cdot)$ converges in the space $\mathcal{C}^0([0,T], H^1_{\rm loc}(\mathbb{R}) \times L^2_{\rm loc}(\mathbb{R}))$ to some limit $(w, \dot w)$ which satisfies (1.1). Note that $|w(0,0)| = \varepsilon_0$, because $|v(\overline{x}(t_n), t_n)| = \varepsilon_0$ for all n by definition of the invasion point.

Let c_-, c_+ be the parabolic speeds associated to s_-, s_+ according to (4.1) (if $s_- \le 0$, we simply take $c_- = 0$). We choose any $c \in (c_-, c_+)$ such that $c > c_+ - \eta$, where η is the positive constant given by Proposition 4.2. By construction, $\overline{y}_c(t_n) \to +\infty$ and $\overline{y}_c(t'_n) \to -\infty$ as $n \to \infty$. Applying (4.13) with $y_0 = t_0 = 0$, we see that there exists $C_8 > 0$ such that $|R_c(0,t)| \le C_8\, e^{-\mu t/2}$ for all $t \ge 0$. Since $\partial_t E_c(0,t) \le R_c(0,t)$, it follows that $E_c(0,t) \ge E_c(0,t'_n) - C_9\, e^{-\mu t/2}$ for $t \in [0, t'_n]$, where $C_9 = 2C_8/\mu$. Taking the limit $n \to \infty$ and using (4.18), we conclude that $E_c(0,t) \ge -C_9\, e^{-\mu t/2}$ for all $t \ge 0$.

On the other hand, using (4.7) and the estimate above on $R_c(0,t)$, we obtain

$$E_c(0,t) - E_c(0,0) \;\le\; -(1+\alpha c^2)\int_0^t \int_{\mathbb{R}} e^{cy} |\dot v_c(y,\tau)|^2 \,\mathrm{d}y\,\mathrm{d}\tau + C_9(1 - e^{-\mu t/2}) \;.$$

Recalling that $E_c(0,t) \ge -C_9 e^{-\mu t/2}$ and setting $t = t_n + T$, we find

$$\begin{aligned} E_c(0,0) \;&\ge\; -C_9 + (1+\alpha c^2)\int_0^{t_n+T} \int_{\mathbb{R}} e^{cy} |\dot v_c(y,t)|^2 \,\mathrm{d}y\,\mathrm{d}t \\ &\ge\; -C_9 + (1+\alpha c^2)\int_{t_n}^{t_n+T} \int_{\mathbb{R}} e^{c(\overline{y}_c(t_n)+y)} |\dot v_c(\overline{y}_c(t_n)+y,t)|^2 \,\mathrm{d}y\,\mathrm{d}t \;, \end{aligned}$$

hence

$$\int_0^T \int_{\mathbb{R}} e^{cy} |\dot v_c(\overline{y}_c(t_n)+y, t_n+t)|^2 \,\mathrm{d}y\,\mathrm{d}t \;\le\; \frac{e^{-c\overline{y}_c(t_n)}}{(1+\alpha c^2)} (E_c(0,0) + C_9) \;\xrightarrow[n\to\infty]{}\; 0 \;. \tag{5.2}$$

Since

$$\dot v_c(y,t) \;=\; \dot v\Big(\frac{y+ct}{\sqrt{1+\alpha c^2}}\,, t\Big) + s v'\Big(\frac{y+ct}{\sqrt{1+\alpha c^2}}\,, t\Big) \;, \quad \text{where} \quad s = \frac{c}{\sqrt{1+\alpha c^2}} \;,$$

it follows from (5.2) that $\int_0^T \int_{-L}^L |\dot v + s v'|^2(\bar x(t_n) + x, t_n + t)\,\mathrm{d}x\,\mathrm{d}t$ converges to zero as $n \to \infty$ for any $L > 0$. Passing to the limit, we conclude that $\dot w(x,t) + s w'(x,t) = 0$ for all $t \in [0,T]$ and (almost) all $x \in \mathbb{R}$. The key point is that this identity must hold for all $c \in (c_-, c_+)$ such that $c > c_+ - \eta$; i.e., for all s in a nonempty open interval. Obviously, this implies that $w' = \dot w = 0$, hence, since $|w(0,0)| = \varepsilon_0$, $w(x,t)$ is identically equal either to ε_0 or to $-\varepsilon_0$. But this is impossible, because w must be a solution of (1.1) and we know from (3.2) that $V'(\pm\varepsilon_0) \ne 0$. □

6 Control of the energy around the invasion point

Proposition 5.1 shows that the invasion point $\overline{x}(t)$ has an average speed $s_\infty \in (0, 1/\sqrt{\alpha})$ as $t \to +\infty$. Our next objective is to prove that the solution $v(x,t)$ of (3.8) converges in any neighborhood of the invasion point to the profile of a travelling front. To achieve this goal, a crucial step is to control the energy $E_c(\overline{y}_c(t), t)$ for c close to c_∞, where

$$c_\infty = \frac{s_\infty}{\sqrt{1 - \alpha s_\infty^2}} . \tag{6.1}$$

The main result of this section is:

Proposition 6.1. *There exists a positive constant η such that, for all $c \in [c_\infty - \eta, c_\infty + \eta]$, the energy $E_c(\overline{y}_c(t), t)$ is a bounded function of $t \geq 0$.*

Remarks:
1. It is important to realize that, unlike in the previous sections, we do not consider in Proposition 6.1 the energy $E_c(y_0, t)$ located at some fixed point y_0 in the moving frame, but the energy $E_c(\overline{y}_c(t), t)$ located *at the invasion point*. From (4.18) we know that $E_c(\overline{y}_c(t), t) \geq (\min V)/c$ for all $t \geq 0$, hence the only problem is to find an upper bound. If c is close to c_∞, it follows from Proposition 4.2 (with $c_+ = c_\infty$) that $E_c(\overline{y}_c(t_0), t)$ is bounded from above for all $t \geq t_0 \geq 0$. Thus, using the relation

$$E_c(\overline{y}_c(t), t) = e^{c(\overline{y}_c(t_0) - \overline{y}_c(t))} E_c(\overline{y}_c(t_0), t) , \quad t \geq t_0 , \tag{6.2}$$

which follows immediately from (4.8), we see that $E_c(\overline{y}_c(t), t)$ is bounded from above if $\overline{y}_c(t)$ stays bounded from below. Unfortunately $\overline{y}_c(t) \to -\infty$ as $t \to +\infty$ if $c > c_\infty$, and even if $c = c_\infty$ we do not know a priori if $\overline{y}_c(t)$ is bounded from below. The essential ingredients in the proof of Proposition 6.1 are Lemma 6.2, which allows to control the growth of the exponential factor $e^{c(\overline{y}_c(t_0) - \overline{y}_c(t))}$ in the right-hand side of (6.2), and Lemma 6.4, which shows that the energy $E_c(\overline{y}_c(t_0), t)$ decays significantly under appropriate conditions.
2. We shall prove in Section 7 that $c_\infty = c_*$ and that the function $v_{c_*}(\overline{y}_{c_*}(t) + \cdot, t)$ converges uniformly on compact sets to the unique solution h of (1.7) such that $h(0) = \varepsilon_0$. Now, it is easy to verify that $h \in H^1_c(\mathbb{R})$ for $c < c_h \equiv \frac{1}{2}(c_* + \sqrt{c_*^2 + 4V''(0)})$. In agreement with Proposition 6.1, we thus expect that the energy $E_c(\overline{y}_c(t), t)$ stays bounded for all times if c is close to $c_\infty = c_*$, and blows up if $c > c_h$.
3. The fact that the conclusion of Proposition 6.1 holds not only for $c = c_\infty$ but for all c in a neighborhood of the invasion speed is one of the key points of our convergence proof. It will allow in Section 7 to control the variation of the energy $E_c(\overline{y}_c(t), t)$ when the parameter c is increased, a difficult task due to the exponential weight e^{cy} in (4.7). This problem is completely avoided in the alternative approach of [38] where only bounded weights are used.

The first step in the proof of Proposition 6.1 consists in showing that the invasion point $\bar{x}(t)$ cannot make arbitrarily large jumps to the left.

Lemma 6.2. *There exists $\eta > 0$ such that, for all $c \in (c_\infty - \eta, c_\infty)$, there exists a positive constant M_c such that,*

$$\overline{y}_c(t') \geq \overline{y}_c(t) - M_c , \quad \textit{for all } t' \geq t \geq 0 . \tag{6.3}$$

Proof: Let η be the positive constant given by Proposition 4.2 for $c_+ = c_\infty$. To prove (6.3), we argue by contradiction. Assume that there exist a speed $c \in (c_\infty - \eta, c_\infty)$ and two sequences of times $\{t_n\}$ and $\{t'_n\}$ such that $t'_n > t_n \geq 0$ for all $n \in \mathbb{N}$ and $\overline{y}_c(t'_n) - \overline{y}_c(t_n) \to -\infty$ as $n \to \infty$. Since $c < c_\infty$, we know that $\overline{y}_c(t) \to +\infty$ as $t \to +\infty$, and therefore we must have $t_n \to +\infty$ as $n \to \infty$. Thus, if we fix any $T > 0$, we can apply Proposition 2.3 and assume without loss of generality that the sequence of functions $(v, \dot{v})(\overline{x}(t_n) + \cdot, t_n + \cdot)$ converges in the space $\mathcal{C}^0([-T, 0], H^1_{\rm loc}(\mathbb{R}) \times L^2_{\rm loc}(\mathbb{R}))$ toward some limit $(w, \dot{w})$ which satisfies (1.1).

Using (4.9) and Proposition 4.2, we obtain for all $n \in \mathbb{N}$:

$$
\begin{aligned}
&(1+\alpha c^2) \int_{t_n - T}^{t'_n} \int_{\mathbb{R}} e^{cy} \left|\dot{v}_c(\overline{y}_c(t_n) + y, t)\right|^2 \, \mathrm{d}y \, \mathrm{d}t \\
&\quad \leq E_c(\overline{y}_c(t_n), t_n - T) - E_c(\overline{y}_c(t_n), t'_n) + \frac{2K_4}{\mu} \max\Big(E_c(\overline{y}_c(t_n), t_n - T) \,,\, e^{-c\overline{y}_c(t_n)}\Big) \,.
\end{aligned}
$$

In view of (6.2), we have

$$
\max\Big(E_c(\overline{y}_c(t_n), t_n - T) \,,\, e^{-c\overline{y}_c(t_n)}\Big) \;=\; e^{-c\overline{y}_c(t_n)} \max(E_c(0, t_n - T) \,, 1) \;\xrightarrow[n\to\infty]{}\; 0 \,,
$$

because $\overline{y}_c(t_n) \to +\infty$ and $E_c(0, t_n - T)$ is bounded from above due to (4.12). On the other hand, since $\overline{y}_c(t'_n) - \overline{y}_c(t_n) \to -\infty$ by assumption, it follows from (4.18) that

$$
-E_c(\overline{y}_c(t_n), t'_n) \;\leq\; e^{c(\overline{y}_c(t'_n) - \overline{y}_c(t_n))} \, \frac{|\min V|}{c} \;\xrightarrow[n\to\infty]{}\; 0 \,.
$$

Thus, we have shown:

$$
(1+\alpha c^2) \int_{-T}^{0} \int_{\mathbb{R}} e^{cy} \left|\dot{v}_c(\overline{y}_c(t_n) + y, t_n + t)\right|^2 \, \mathrm{d}y \, \mathrm{d}t \;\xrightarrow[n\to\infty]{}\; 0 \,.
$$

Proceeding as in the proof of Lemma 5.4, we conclude that $\dot{w}(x,t) + s w'(x,t) = 0$ for all $t \in [-T, 0]$ and (almost) all $x \in \mathbb{R}$, where $s = c/\sqrt{1+\alpha c^2}$. Now, the crucial observation is that, for any $c' \in (c, c_\infty)$, we still have $\overline{y}_{c'}(t_n) \to +\infty$ and $\overline{y}_{c'}(t'_n) - \overline{y}_{c'}(t_n) \to -\infty$ as $n \to \infty$. The second claim follows immediately from the identity

$$
\overline{y}_{c_2}(t') - \overline{y}_{c_2}(t) \;=\; \frac{\sqrt{1+\alpha c_2^2}}{\sqrt{1+\alpha c_1^2}} \Big(\overline{y}_{c_1}(t') - \overline{y}_{c_1}(t)\Big) + \Big(\frac{\sqrt{1+\alpha c_2^2}}{\sqrt{1+\alpha c_1^2}}\, c_1 - c_2\Big)(t' - t) \,. \tag{6.4}
$$

Thus, repeating the same arguments, we conclude that $\dot{w} + s' w' = 0$ for all s' in a nonempty open interval. This implies that $\dot{w} = w' = 0$, and we obtain a contradiction as in Lemma 5.4. □

Combining the bound (6.3) and the identity (6.4), we obtain the following useful estimate, which is valid in any frame whose speed is close enough to the invasion speed.

Corollary 6.3. *For all $p > 0$, there exist $\eta > 0$ and $M > 0$ such that, for all $c \in [c_\infty - \eta, c_\infty + \eta]$, one has*

$$
\overline{y}_c(t') - \overline{y}_c(t) \;\geq\; -M - p(t' - t) \,, \quad \textit{for all } t' \geq t \geq 0 \,. \tag{6.5}
$$

The next proposition shows that, if the energy is sufficiently large at a given time, and if the invasion point stays bounded from above on a sufficiently long time interval, then a significant decay of energy must occur.

Lemma 6.4. *There exist positive constants η, t_0, ω, and K_5 such that the following holds. For any $c \in [c_\infty - \eta, c_\infty + \eta]$, any $y_0 \in \mathbb{R}$, any $t_1 \geq t_0$, any $T > 0$, and any $M > 0$, if the invasion point satisfies $\overline{y}_c(t) \leq y_0 + M$ for all $t \in [t_1, t_1 + T]$, then*

$$E_c(y_0, t_1 + T) \;\leq\; K_5\Big(e^{-\omega T}\, E_c(y_0, t_1) + e^{cM}\Big)\ . \tag{6.6}$$

Proof: Let η be the positive number given by Proposition 4.2 for $c_+ = c_\infty$ and let $c \in [c_\infty - \eta, c_\infty + \eta]$. Given $y_0 \in \mathbb{R}$, we define

$$\mathcal{E}_c(y_0, t) \;=\; \int_{\mathbb{R}} e^{cy}\Big(\frac{\alpha}{2}|\dot v_c|^2 + \frac{1}{2}|v_c'|^2 + V(v_c) + \alpha\gamma v_c \dot v_c\Big)(y_0 + y, t)\,\mathrm{d}y\ ,$$

where $\gamma > 0$ will be fixed later. Equation (4.4) satisfied by v_c implies that

$$\begin{aligned}\partial_t \mathcal{E}_c(y_0, t) = \int_{\mathbb{R}} e^{cy}\Big(&-(1+\alpha c^2)|\dot v_c|^2 + f(v_c, r_c)\dot v_c + \alpha\gamma|\dot v_c|^2 - \gamma(1+2\alpha c^2)v_c\dot v_c \\ &- 2\alpha\gamma c v_c'\dot v_c - \gamma|v_c'|^2 - \gamma V'(v_c)v_c + \gamma f(v_c, r_c)v_c\Big)(y_0 + y, t)\,\mathrm{d}y\ .\end{aligned}$$

From (3.11) and (4.3) we know that

$$\int_{\mathbb{R}} e^{cy}|f(v_c, r_c)|(|\dot v_c| + \gamma|v_c|)(y_0 + y, t)\,\mathrm{d}y \;\leq\; K_2\varepsilon_0\, e^{-\mu t}\int_{\mathbb{R}} e^{cy}(|v_c|^2 + |\dot v_c|^2)(y_0 + y, t)\,\mathrm{d}y\ ,$$

provided that $\gamma \leq 3/4$. Thus, using the bound $2ab \leq C^{-1}a^2 + Cb^2$ and the Poincaré inequality (4.15), we obtain that, if γ is small enough and t_0 is large enough, the following estimate holds for all $t \geq t_0$:

$$\partial_t \mathcal{E}_c(y_0, t) \;\leq\; -\int_{\mathbb{R}} e^{cy}\Big(\frac{1}{2}|\dot v_c|^2 + \frac{\gamma}{2}|v_c'|^2 + \gamma V'(v_c)v_c\Big)(y_0 + y, t)\,\mathrm{d}y\ .$$

On the other hand, we know from (4.3) that $|v_c(y,t)|$ is uniformly bounded for all $y \in \mathbb{R}$ and all $t \geq 0$, and from (3.2) that $2V'(v_c(y,t))v_c(y,t) \geq V(v_c(y,t)) \geq (\beta_0/4)v_c(y,t)^2$ for all $y \geq \overline{y}_c(t)$. Thus, there exist $\omega > 0$ and $C_0 > 0$ such that

$$\begin{aligned}\partial_t \mathcal{E}_c(y_0, t) \;&\leq\; -\omega\mathcal{E}_c(y_0, t) + C\int_{-\infty}^{\overline{y}_c(t) - y_0} e^{cy}\Big(|V'(v_c)v_c| + |V(v_c)|\Big)(y_0 + y, t)\,\mathrm{d}y\ ,\\ &\leq\; -\omega\mathcal{E}_c(y_0, t) + C_0\, e^{c(\overline{y}_c(t) - y_0)}\ ,\end{aligned} \tag{6.7}$$

for all $t \geq t_0$. In a similar way, there exist $C_1 > 1$ and $C_2 > 0$ such that

$$C_1^{-1}E_c(y_0, t) - C_2\, e^{c(\overline{y}_c(t) - y_0)} \;\leq\; \mathcal{E}_c(y_0, t) \;\leq\; C_1 E_c(y_0, t) + C_2\, e^{c(\overline{y}_c(t) - y_0)}\ , \tag{6.8}$$

for all $t \geq t_0$. Remark that, in (6.7) and (6.8), all constants can be chosen to be independent of $y_0 \in \mathbb{R}$, of $c \in [c_\infty - \eta, c_\infty + \eta]$, and of $t \geq t_0$.

Now, we fix $t_1 \geq t_0$ and assume that $\overline{y}_c(t) - y_0 \leq M$ when $t \in [t_1, t_1 + T]$, for some $M > 0$ and some $T > 0$. Integrating the differential inequality (6.7), we find

$$\mathcal{E}_c(y_0, t_1 + T) \;\leq\; e^{-\omega T}\mathcal{E}_c(y_0, t_1) + \frac{C_0}{\omega}\, e^{cM} \;.$$

Combining this result with (6.8), we arrive at

$$E_c(y_0, t_1 + T) \;\leq\; C_1^2\, e^{-\omega T} E_c(y_0, t_1) + \Big(2C_1C_2 + \frac{C_0C_1}{\omega}\Big)\, e^{cM} \;,$$

which is the desired estimate. □

Using the control on the invasion point given by Lemma 6.2 and the decay of energy described in Lemma 6.4, we are now able to prove the main result.

Proof of Proposition 6.1: Let t_0, ω, K_5 be as in Lemma 6.4, and choose $p > 0$ such that $4c_\infty p \leq \min(\omega, \mu)$, where $\mu > 0$ is as in Lemma 4.1. By Corollary 6.3, there exist $\eta > 0$ and $M > 0$ such that (6.5) holds, and without loss of generality we can assume that $\eta < c_\infty$ and that M is large enough so that $e^{-(c_\infty - \eta)M} \leq 1/2$. In the rest of the proof, we fix some $c \in [c_\infty - \eta, c_\infty + \eta] \subset (0, 2c_\infty)$ (but all constants will be independent of c).

From (4.9) and Proposition 4.2 we know that, for all $t \geq 0$ and all $\tau \geq 0$,

$$\begin{aligned}
\partial_\tau E_c(\overline{y}_c(t), t + \tau) \;&\leq\; K_4\, e^{-\mu(t+\tau)/2}\, \max\Big(E_c(\overline{y}_c(t), t)\,,\, e^{-c\overline{y}_c(t)}\Big) \\
&\leq\; K_4\, e^{-\mu(t+\tau)/2}\, e^{-c\overline{y}_c(t)}\, \max\Big(E_c(0, t)\,,\, 1\Big) \;.
\end{aligned}$$

Since, by (6.5), $c\overline{y}_c(t) \geq c(\overline{y}_c(0) - M - pt) \geq -C - \mu t/2$ for all $t \geq 0$, and since $E_c(0, t)$ is uniformly bounded from above by Proposition 4.2, there exists $C_3 > 0$ such that, for all $t \geq 0$ and all $\tau \geq 0$,

$$E_c(\overline{y}_c(t), t + \tau) \;\leq\; E_c(\overline{y}_c(t), t) + C_3 \;. \tag{6.9}$$

Now, we choose $T \geq 1$ and $C_4 \geq C_3$ such that the following inequalities hold:

$$4K_5\, e^{cM}\, e^{-\omega T/2} \;\leq\; 1 \;, \quad \text{and} \quad 4K_5\, e^{4cM}\, e^{cp(1+T)} \;\leq\; C_4 \;.$$

We claim that, if $E_c(\overline{y}_c(t), t) \geq C_4$ for some $t \geq t_0$, then there exists $t' \in [1, T]$ such that $E_c(\overline{y}_c(t + t'), t + t') \leq \frac{1}{2} E_c(\overline{y}_c(t), t)$.

To prove this claim, we distinguish two possible cases. If there exists $\tau \in [0, T]$ such that $\overline{y}_c(t + \tau) \geq \overline{y}_c(t) + 3M + p$, then $t' = \max(\tau, 1)$ is a suitable choice. Indeed, by Corollary 6.3, we have $\overline{y}_c(t + t') \geq \overline{y}_c(t) + 2M$ and so, using (6.9), we find

$$\begin{aligned}
E_c(\overline{y}_c(t + t'), t + t') \;&=\; e^{c(\overline{y}_c(t) - \overline{y}_c(t+t'))} E_c(\overline{y}_c(t), t + t') \\
&\leq\; e^{-2cM} (E_c(\overline{y}_c(t), t) + C_3) \;\leq\; \tfrac{1}{2} E_c(\overline{y}_c(t), t) \;,
\end{aligned}$$

because $e^{-2cM} \leq 1/4$ and $C_3 \leq C_4 \leq E_c(\overline{y}_c(t), t)$ by assumption. On the other hand, if $\overline{y}_c(t + \tau) \leq \overline{y}_c(t) + 3M + p$ for all $\tau \in [0, T]$, we can take $t' = T$ because, due to Corollary 6.3, Lemma 6.4 and our choices of T, C_4, and p, we have

$$\begin{aligned}
E_c(\overline{y}_c(t + T), t + T) \;&=\; e^{c(\overline{y}_c(t) - \overline{y}_c(t+T))} E_c(\overline{y}_c(t), t + T) \\
&\leq\; e^{c(M+pT)} K_5 \Big(e^{-\omega T} E_c(\overline{y}_c(t), t) + e^{c(3M+p)}\Big) \;\leq\; \tfrac{1}{2} E_c(\overline{y}_c(t), t) \;.
\end{aligned}$$

We now show that the claim above implies Proposition 6.1. To this purpose, we construct the following sequence of times. We take $t_0 > 0$ as in Lemma 6.4, and given t_n we define t_{n+1} in the following way. If $E_c(\overline{y}_c(t_n), t_n) \le C_4$, we simply set $t_{n+1} = t_n + 1$ and, using (6.9) and Corollary 6.3, we get

$$E_c(\overline{y}_c(t_{n+1}), t_{n+1}) \;=\; e^{c(\overline{y}_c(t_n)-\overline{y}_c(t_{n+1}))} E_c(\overline{y}_c(t_n), t_{n+1}) \;\le\; e^{c(M+p)}(C_4 + C_3)\ .$$

If $E_c(\overline{y}_c(t_n), t_n) \ge C_4$ we set $t_{n+1} = t_n + t'$, where $t' \ge 1$ is the time given by the claim above when $t = t_n$. In this case, we know that $E_c(\overline{y}_c(t_{n+1}), t_{n+1}) \le \frac{1}{2} E_c(\overline{y}_c(t_n), t_n)$. By contruction, the sequence $\{t_n\}$ goes to $+\infty$ as $n \to \infty$, and

$$\limsup_{n\to\infty} E_c(\overline{y}_c(t_n), t_n) \;\le\; e^{c(M+p)}(C_4 + C_3)\ .$$

To conclude the proof, we observe that Eq. (6.9) and Corollary 6.3 provide a control on the energy for the remaining times. Indeed, since $t_{n+1} - t_n \le T$ for all n, we have for all $t \in [t_n, t_{n+1}]$ $\overline{y}_c(t) \ge \overline{y}_c(t_n) - M - pT$ and hence, for all $t \in [t_n, t_{n+1}]$,

$$E_c(\overline{y}_c(t), t) \;\le\; e^{c(M+pT)} \max(E_c(\overline{y}_c(t_n), t)\ , 0) \;\le\; e^{c(M+pT)} \max(E_c(\overline{y}_c(t_n), t_n) + C_3\ , 0)\ .$$

A similar argument shows that $E_c(\overline{y}_c(t), t)$ is bounded from above for $t \in [0, t_0]$. □

7 Convergence to a travelling wave

The purpose of this section is to show that, for any $L > 0$, the solution v of (3.8) converges to a travelling front uniformly in the interval $(\overline{x}(t) - L, +\infty)$. The key step is to prove that, in the frame moving at the invasion speed s_∞, the energy dissipation around the invasion point converges to zero as $t \to +\infty$.

Proposition 7.1. *Let s_∞ be the invasion speed introduced in Proposition 5.1 and let c_∞ be the parabolic speed* (6.1)*. For any $T > 0$, we have*

$$\int_{t-T}^{t}\int_{\mathbb{R}} e^{c_\infty y}|\dot v_{c_\infty}(\overline{y}_{c_\infty}(t) + y, \tau)|^2\, \mathrm{d}y\, \mathrm{d}\tau \;\xrightarrow[t\to+\infty]{}\; 0\ .$$

We start the proof with an auxiliary result showing that the energy $E_c(\overline{y}_c(t), t)$ is a continuous function of the parameter c.

Lemma 7.2. *Let $\eta > 0$ be as in Proposition 6.1. Given any $T \ge 0$, there exists $K_6 > 0$ such that, for all $c_1, c_2 \in [c_\infty - \eta/2, c_\infty + \eta/2]$, all $t \ge T$ and all $\tau \in [t-T, t]$, the following estimate holds:*

$$|E_{c_1}(\overline{y}_{c_1}(t), \tau) - E_{c_2}(\overline{y}_{c_2}(t), \tau)| \;\le\; K_6 |c_1 - c_2|\ .$$

Proof: If we return to the original variables using the definitions (4.2), (4.5), we obtain the identity

$$\begin{aligned} E_c(\overline{y}_c(t), \tau) \;&=\; \int_{\mathbb{R}} e^{cy}\Big(\frac{\alpha}{2}|\dot v_c|^2 + \frac{1}{2}|v_c'|^2 + V(v_c)\Big)(\overline{y}_c(t) + y, \tau)\, \mathrm{d}y \qquad (7.1)\\ &=\; \sqrt{1{+}\alpha c^2}\, e^{c(\overline{y}_c(\tau)-\overline{y}_c(t))} \int_{\mathbb{R}} e^{c\sqrt{1+\alpha c^2}x}\Big(\frac{\alpha}{2}|\dot v{+}sv'|^2 + \frac{|v'|^2}{2(1{+}\alpha c^2)} + V(v)\Big)(\overline{x}(\tau){+}x, \tau)\, \mathrm{d}x\,, \end{aligned}$$

where $s = c/\sqrt{1+\alpha c^2}$. Assume first that $\tau = t$ and $c = \overline{c}$, where $\overline{c} = c_\infty + \eta$. We know from Proposition 6.1 that $E_{\overline{c}}(\overline{y}_{\overline{c}}(t), t)$ is bounded (from above) for all times. On the other hand, we obtain a lower bound on the last member of (7.1) if we replace $V(v(\overline{x}(t)+x,t))$ by zero if $x \ge 0$ and by $\min V$ if $x \le 0$. Thus, using in addition the Poincaré inequality (4.15) to control $|v|^2$ in terms of $|v'|^2$, we deduce from (7.1) that there exists a constant $C_0 > 0$ such that

$$\int_{\mathbb{R}} e^{\overline{c}\sqrt{1+\alpha\overline{c}^2}x}\Big(|\dot{v}|^2 + |v'|^2 + |v|^2\Big)(\overline{x}(t)+x,t)\,\mathrm{d}x \;\le\; C_0\;, \quad \text{for all } t \ge 0\;. \tag{7.2}$$

Using the uniform control (7.2), it is a straightforward exercise to verify that the last member of (7.1) is indeed a Lipschitz function of $c \in [c_\infty - \eta/2, c_\infty + \eta/2]$, uniformly in $t \ge T$ and $\tau \in [t-T, t]$. The only potential difficulty comes from the exponential terms. If we denote

$$Y_c(t,\tau) \;=\; c(\overline{y}_c(\tau) - \overline{y}_c(t)) \;=\; c\sqrt{1+\alpha c^2}\,(\overline{x}(\tau) - \overline{x}(t)) + c^2(t-\tau)\;,$$

we know from Corollary 6.3 that $Y_c(t,\tau) \le c(M+pT)$ for all $\tau \in [t-T,t]$ and all $c \in [c_\infty - \eta, c_\infty + \eta]$, hence

$$\Big|e^{Y_{c_1}(t,\tau)} - e^{Y_{c_2}(t,\tau)}\Big| \;\le\; e^{\max(Y_{c_1}(t,\tau),Y_{c_2}(t,\tau))}\,|Y_{c_1}(t,\tau) - Y_{c_2}(t,\tau)| \;\le\; C_1|c_1 - c_2|\;.$$

On the other hand, if $c_1, c_2 \in [c_\infty - \eta/2, c_\infty + \eta/2]$, we can bound

$$\Big|e^{c_1\sqrt{1+\alpha c_1^2}x} - e^{c_2\sqrt{1+\alpha c_2^2}x}\Big| \;\le\; C_2|c_1 - c_2|\begin{cases} e^{\overline{c}\sqrt{1+\alpha\overline{c}^2}x} & \text{if} \quad x \ge 0\;, \\ e^{\underline{c}\sqrt{1+\alpha\underline{c}^2}x} & \text{if} \quad x \le 0\;, \end{cases}$$

where $\overline{c} = c_\infty + \eta$ and $\underline{c} = c_\infty - \eta$. Thus, using estimate (7.2) for $x \ge 0$ and the uniform bound (3.12) for $x \le 0$, we obtain the desired conclusion. □

Proof of Proposition 7.1: We argue by contradiction. Assume that there exist $\hat{\delta} > 0$, $T > 0$, and a sequence of times $\{t_n\}$ going to $+\infty$ such that

$$\int_{t_n - T}^{t_n}\int_{\mathbb{R}} e^{c_\infty y}\,|\dot{v}_{c_\infty}(\overline{y}_{c_\infty}(t_n) + y, t)|^2\,\mathrm{d}y\,\mathrm{d}t \;\ge\; \hat{\delta}\;, \tag{7.3}$$

for all $n \in \mathbb{N}$. Following an idea introduced in [38], we shall arrive at a contradiction by considering the variation of the energy along the *broken line* connecting the points $(t_n, \overline{y}_{c_\infty}(t_n))$ in the (t,y) plane, see Fig. 2.

To this end we define, for all $n \in \mathbb{N}$,

$$s_n \;=\; \frac{1}{\sqrt{1+\alpha c_\infty^2}}\frac{\overline{y}_{c_\infty}(t_{n+1}) - \overline{y}_{c_\infty}(t_n)}{t_{n+1} - t_n} + s_\infty\;, \quad \text{and} \quad c_n \;=\; \frac{s_n}{\sqrt{1-\alpha s_n^2}}\;.$$

As is easily verified, the speed s_n is the slope of the line segment connecting $(t_n, \overline{x}(t_n))$ and $(t_{n+1}, \overline{x}(t_{n+1}))$; i.e., $\overline{x}(t_{n+1}) - \overline{x}(t_n) = s_n(t_{n+1} - t_n)$. Since $\overline{y}_{c_\infty}(t)/t$ converges to zero as $t \to +\infty$ by Proposition 5.1, we can assume (up to extracting a subsequence) that

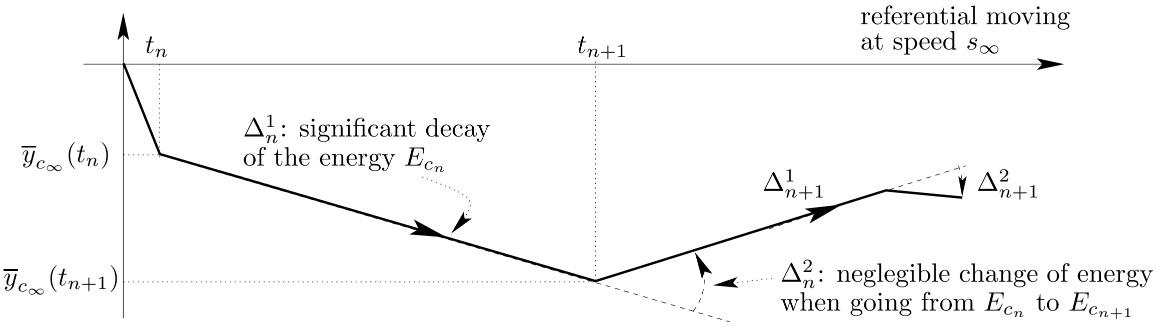

Fig. 2: A schematic picture of the broken line in the (t, y)-plane which is used in the proof of Proposition 7.1. The main idea is to control the energy decay along the straight sections, as well as the small energy variations occuring at the vertices.

$s_n \in (0, 1/\sqrt{\alpha})$ for all $n \in \mathbb{N}$ and that $s_n \to s_\infty$ as $n \to \infty$. Then the parabolic speed c_n is well-defined for all n, and by construction $\overline{y}_{c_n}(t_{n+1}) = \overline{y}_{c_n}(t_n)$. Extracting another subsequence if needed, we can further assume that $t_{n+1} \geq t_n + T$ and $|c_n - c_\infty| \leq \eta/2$ for all $n \in \mathbb{N}$, where $\eta > 0$ is as in Proposition 6.1, and also assume that the sum $\sum_{n\geq 0} |c_n - c_\infty|$ is finite.

Now we define, for each $n \in \mathbb{N}$,

$$\begin{aligned}\Delta_n &= E_{c_n}(\overline{y}_{c_n}(t_n), t_n) - E_{c_{n+1}}(\overline{y}_{c_{n+1}}(t_{n+1}), t_{n+1}) \\ &= E_{c_n}(\overline{y}_{c_n}(t_n), t_n) - E_{c_n}(\overline{y}_{c_n}(t_{n+1}), t_{n+1}) \\ &+ E_{c_n}(\overline{y}_{c_n}(t_{n+1}), t_{n+1}) - E_{c_{n+1}}(\overline{y}_{c_{n+1}}(t_{n+1}), t_{n+1}) \;=\; \Delta_n^1 + \Delta_n^2 \;.\end{aligned}$$

Since $\overline{y}_{c_n}(t_{n+1}) = \overline{y}_{c_n}(t_n)$, the quantity Δ_n^1 is the variation of the energy $E_{c_n}(y_0, t)$ at a fixed point $y_0 \in \mathbb{R}$ on the time interval $[t_n, t_{n+1}]$. By (4.9) and Proposition 4.2, we have

$$\Delta_n^1 \;=\; (1 + \alpha c_n^2) \int_{t_n}^{t_{n+1}} \int_{\mathbb{R}} e^{c_n y} |\dot{v}_{c_n}(\overline{y}_{c_n}(t_{n+1}) + y, t)|^2 \,\mathrm{d}y \,\mathrm{d}t - \int_{t_n}^{t_{n+1}} R_{c_n}(\overline{y}_{c_n}(t_n), t) \,\mathrm{d}t \;,$$

and

$$|R_{c_n}(\overline{y}_{c_n}(t_n), t)| \;\leq\; K_4 \, e^{-\mu t/2} \, e^{-c_n \overline{y}_{c_n}(t_n)} \, \max(E_{c_n}(0, t_n)\,,\, 1) \;.$$

But $E_{c_n}(0, t_n)$ is bounded by from above uniformly in n by Proposition 4.2, and since $\overline{y}_{c_n}(t_n)/t_n$ converges to zero as $n \to \infty$ we can assume without loss of generality that $c_n \overline{y}_{c_n}(t_n) \geq -\mu t_n/4$ for all $n \in \mathbb{N}$. Thus

$$\Delta_n^1 \;\geq\; \int_{t_{n+1}-T}^{t_{n+1}} \int_{\mathbb{R}} e^{c_n y} |\dot{v}_{c_n}(\overline{y}_{c_n}(t_{n+1}) + y, t)|^2 \,\mathrm{d}y \,\mathrm{d}t - C_3 \, e^{-\mu t_n/4} \;,$$

for some $C_3 > 0$. Moreover, since $|c_n - c_\infty| \leq \eta/2$, the proof of Lemma 7.2 shows that, for all $t \in [t_{n+1} - T, t_{n+1}]$,

$$\left| \int_{\mathbb{R}} e^{c_n y} |\dot{v}_{c_n}(\overline{y}_{c_n}(t_{n+1}) + y, t)|^2 \,\mathrm{d}y \,\mathrm{d}t - \int_{\mathbb{R}} e^{c_\infty y} |\dot{v}_{c_\infty}(\overline{y}_{c_\infty}(t_{n+1}) + y, t)|^2 \,\mathrm{d}y \,\mathrm{d}t \right| \;\leq\; C_4 |c_n - c_\infty| \,,$$

for some $C_4 > 0$. Combining both estimates and using the assumption (7.3), we thus obtain

$$\Delta_n^1 \;\ge\; \hat{\delta} - C_4 T|c_n - c_\infty| - C_3\, e^{-\mu t_n/4}\;, \quad n \in \mathbb{N}\;.$$

On the other hand, the quantity Δ_n^2 represents the change in the energy $E_c(\overline{y}_c(t_{n+1}), t_{n+1})$ when c varies from c_n to c_{n+1}. By Lemma 7.2, we have $|\Delta_n^2| \le K_6|c_n - c_{n+1}|$, hence

$$\Delta_n \;=\; \Delta_n^1 + \Delta_n^2 \;\ge\; \hat{\delta} - K_6|c_n - c_{n+1}| - C_4 T|c_n - c_\infty| - C_3\, e^{-\mu t_n/4}\;, \quad n \in \mathbb{N}\;.$$

To conclude the proof, we observe that

$$E_{c_0}(\overline{y}_{c_0}(t_0), t_0) - E_{c_N}(\overline{y}_{c_N}(t_N), t_N) \;=\; \sum_{n=0}^{N-1} \Delta_n \;\xrightarrow[N\to\infty]{}\; +\infty\;,$$

because $t_n \ge nT$ and the sum $\sum_{n\ge 0}|c_n - c_\infty|$ is finite. Thus $E_{c_N}(\overline{y}_{c_N}(t_N), t_N) \to -\infty$ as $N \to \infty$, in contradiction with the lower bound (4.18). Thus (7.3) cannot hold for all $n \in \mathbb{N}$, and Proposition 7.1 is proved. $\square$

Using now, for the first time, the fact that the differential equation (1.7) has a front-like solution for a single value of c_* and that this solution is unique up to translations, we can establish the local convergence to a travelling front.

Corollary 7.3. *The invasion speed satisfies $s_\infty = s_* \equiv c_*/\sqrt{1+\alpha c_*^2}$, and we have*

$$\int_{\mathbb{R}} e^{c_*\sqrt{1+\alpha c_*^2}\,x}\Big(|\dot{v}(\overline{x}(t)+x,t) + s_* v_*'(x)|^2 + |v'(\overline{x}(t)+x,t) - v_*'(x)|^2 + |v(\overline{x}(t)+x,t) - v_*(x)|^2\Big)\,\mathrm{d}x \;\xrightarrow[t\to+\infty]{}\; 0\;,$$

where $v_*(x) = h(\sqrt{1+\alpha c_*^2}\,x)$ and h is the solution of (1.7) *normalized so that $h(0) = \varepsilon_0$. In particular, v converges to a front uniformly in any interval of the type $(\overline{x}(t) - L, +\infty)$.*

Proof: Fix $T > 0$ and let $\{t_n\}$ be a sequence of times going to $+\infty$ as $n \to \infty$. In view of Proposition 2.3, we can assume that the sequence of functions $(v,\dot{v})(\overline{x}(t_n) + \cdot, t_n + \cdot)$ converges in the space $\mathcal{C}^0([-T,0], H^1_{\mathrm{loc}}(\mathbb{R}) \times L^2_{\mathrm{loc}}(\mathbb{R}))$ to some limit $(w,\dot{w})$ which satisfies (1.1). By (4.2) and Proposition 7.1, we have

$$\int_{-T}^0\int_{\mathbb{R}} e^{c_\infty\sqrt{1+\alpha c_\infty^2}\,x}|\dot{v} + s_\infty v'|^2(\overline{x}(t_n)+x, t_n+t)\,\mathrm{d}x\,\mathrm{d}t \;\xrightarrow[n\to\infty]{}\; 0\;,$$

hence $\dot{w}(x,t) + s_\infty w'(x,t) = 0$ for all $t \in [-T,0]$ and (almost) all $x \in \mathbb{R}$. Setting $w(x,t) = h(\sqrt{1+\alpha c_\infty^2}\,x - c_\infty t)$, we see that h is a solution of the differential equation $h'' + c_\infty h' - V'(h) = 0$. We also know that $|h(x)| \le M_0$ for all $x \le 0$, that $|h(x)| \le \varepsilon_0$ for all $x \ge 0$, and that $|h(0)| = |w(0,0)| = \varepsilon_0$. Due to our assumptions (1.3)–(1.6) on the potential V, these properties together imply that $c_\infty = c_*$, hence $s_\infty = s_*$, and that h is the unique solution of (1.7) such that $h(0) = \varepsilon_0$, see e.g. [2]. Since the limit is unique, we deduce that the convergence above holds in fact for any sequence $t_n \to +\infty$. In particular, if we denote $v_*(x) = h(\sqrt{1+\alpha c_*^2}\,x)$, we conclude that $(v,\dot{v})(\overline{x}(t)+\cdot,t)$ converges as $t \to +\infty$ to $(v_*, -s_* v_*')$ in $H^1([-L,L]) \times L^2([-L,L])$, for any $L > 0$. Using in addition the estimate (7.2) for $x \geq L$, and the uniform bound (3.12) for $x \leq -L$, we obtain the desired conclusion. □

One can extract from the proof of Corollary 7.3 the following useful information on the invasion point:

Lemma 7.4. *For any $T > 0$ we have*

$$\sup_{|\tau| \leq T} |\overline{x}(t+\tau) - \overline{x}(t) - s_*\tau| \xrightarrow[t\to+\infty]{} 0 \ .$$

Proof: Fix $T > 0$, and choose $L > 0$ large enough so that

$$h(\sqrt{1+\alpha c_*^2}\, L) \leq \frac{\varepsilon_0}{2} \ , \quad \text{and} \quad h(-\sqrt{1+\alpha c_*^2}\, L + c_*T) \geq \frac{1+\varepsilon_0}{2} \ , \tag{7.4}$$

where h is as in Corollary 7.3. We claim that, for any $\hat{\delta} > 0$, there exists $t_0 \geq T$ such that, for all $t \geq t_0$,

$$\sup_{\tau\in[-T,0]} \sup_{x\geq -L} |v(\overline{x}(t)+x, t+\tau) - h(\sqrt{1+\alpha c_*^2}\, x - c_*\tau)| \leq \hat{\delta} \ . \tag{7.5}$$

Indeed, if we restrict the values of x to a bounded interval $I = [-L, L']$, where $L' > 0$, the analog of (7.5) follows immediately from the proof of Corollary 7.3 and the fact that $H^1(I) \hookrightarrow L^\infty(I)$. On the other hand, by Lemma 6.2, there exists $C > 0$ such that $\overline{x}(t) \geq \overline{x}(t+\tau) - C$ for all $\tau \in [-T, 0]$. In view of (7.2) we thus have

$$\begin{aligned} &\sup_{\tau\in[-T,0]} \sup_{x\geq L'} |v(\overline{x}(t)+x, t+\tau) - h(\sqrt{1+\alpha c_*^2}\, x - c_*\tau)| \\ &\qquad \leq \sup_{\tau\in[-T,0]} \sup_{x\geq L'-C} |v(\overline{x}(t+\tau)+x, t+\tau)| + h(\sqrt{1+\alpha c_*^2}\, L') \xrightarrow[L'\to+\infty]{} 0 \ , \end{aligned}$$

uniformly in t. This proves (7.5).

We now assume that $\hat{\delta} < \min(\varepsilon_0, 1-\varepsilon_0)/2$. For any $t \geq t_0$ and any $\tau \in [-T, 0]$, it follows from (7.4), (7.5) that

$$|v(\overline{x}(t) - L, t+\tau)| > \varepsilon_0 \ , \quad \text{and} \quad \sup_{x\geq L} |v(\overline{x}(t)+x, t+\tau)| < \varepsilon_0 \ .$$

By the definition (3.13) of the invasion point, this means that $\overline{x}(t+\tau) \in [\overline{x}(t)-L, \overline{x}(t)+L]$. Using (7.5) with $x = \overline{x}(t+\tau) - \overline{x}(t)$ and recalling that $v(\overline{x}(t+\tau), t+\tau) = h(0) = \varepsilon_0$, we obtain

$$\hat{\delta} \geq |\varepsilon_0 - h(\sqrt{1+\alpha c_*^2}\,(\overline{x}(t+\tau) - \overline{x}(t)) - c_*\tau)| \geq m\sqrt{1+\alpha c_*^2}\, |\overline{x}(t+\tau) - \overline{x}(t) - s_*\tau| \ ,$$

where

$$m = \min\Big\{ |h'(y)| \, \Big| \, -\sqrt{1+\alpha c_*^2}\, L \leq y \leq \sqrt{1+\alpha c_*^2}\, L + c_*T \Big\} > 0 \ .$$

Thus $|\overline{x}(t+\tau) - \overline{x}(t) - s_*\tau| \leq \hat{\delta}/(m\sqrt{1+\alpha c_*^2})$ for all $t \geq t_0$ and all $\tau \in [-T, 0]$. Since $\hat{\delta} > 0$ was arbitrarily small, we obtain the desired conclusion. □

8 Repair behind the front

As in the previous sections, we denote by $u(x,t)$ a solution of (1.1) whose initial data fulfill the conditions (1.9), (1.10), where $\delta \le \min(\delta_0,\delta_1)/2$ is small enough so that (5.1) holds. We know from Corollary 7.3 that, for any given $L > 0$, $u(x,t)$ converges to a travelling front uniformly for $x \in (\overline{x}(t) - L, +\infty)$. To conclude the proof of Theorem 1.1, it remains to prove that $u(x,t)$ converges uniformly to 1 far behind the invasion point. Following again the ideas introduced in [38], we shall do this using a suitable energy estimate in the *laboratory frame*.

Proposition 8.1. *There exists a sequence $t_n \to +\infty$ such that*

$$\|u(\overline{x}(t_n) + \cdot, t_n) - v_*\|_{H^1_{\mathrm{ul}}} + \|\dot u(\overline{x}(t_n) + \cdot, t_n) + s_* v_*'\|_{L^2_{\mathrm{ul}}} \xrightarrow[n\to\infty]{} 0 \ , \tag{8.1}$$

where v_* is as in Corollary 7.3.

Remark: Using an additional argument as in [38, Section 9.6], one can show that (8.1) holds in fact for *all* sequences $t_n \to +\infty$. In our case, this follows from the local stability of the travelling front which will be established in the last section.

Proof: We recall that the solution of (1.1) has been decomposed as $u(x,t) = v(x,t) + r(x,t)$, where the remainder $(r,\dot r)$ converges exponentially to zero as $t \to +\infty$ in the uniformly local energy space $X = H^1_{\mathrm{ul}}(\mathbb{R}) \times L^2_{\mathrm{ul}}(\mathbb{R})$. Using this remark and Corollary 7.3, we can construct a sequence of times $\{t_n\}$ satisfying $t_{n+1} \ge t_n + n + 1$ for all $n \in \mathbb{N}$ and such that, for all $t \ge t_n$,

$$\begin{aligned} \sup_{z\ge -2n} \int_z^{z+1} \Big(|\dot u(\overline{x}(t) + x, t) + s_* v_*'(x)|^2 + |u'(\overline{x}(t) + x, t) - v_*'(x)|^2 \\ + |u(\overline{x}(t) + x, t) - v_*(x)|^2 \Big) \,\mathrm{d}x \ \le\ \frac{1}{n+1} \ . \end{aligned} \tag{8.2}$$

Without loss of generality, we also assume that $t_0 \ge 1$. Let $\theta : \mathbb{R} \to [0,1]$ be a smooth, nondecreasing function satisfying $\theta(x) = 0$ for $x \le -1$, $\theta(x) = 1$ for $x \ge 1$, and $\int_{-1}^1 \theta(x)\,\mathrm{d}x = 1$. We define a smooth map $x_+ : [t_0, +\infty) \to \mathbb{R}$ in the following way. For all $n \in \mathbb{N}$, we set

$$x_+(t) \ =\ \int_0^\infty \theta'(t-\tau)\overline{x}(\tau)\,\mathrm{d}\tau - n - \theta\Big(\frac{2t - t_n - t_{n+1}}{t_{n+1} - t_n}\Big) \ , \quad \text{if } t \in [t_n, t_{n+1}] \ .$$

We recall that $t \mapsto \overline{x}(t)$ is upper semi-continuous, hence measurable. Since $\int_{\mathbb{R}} \theta'(x)\,\mathrm{d}x = 1$ and $\int_{\mathbb{R}} x\theta'(x)\,\mathrm{d}x = 0$, we have for all $t \ge 1$:

$$\overline{x}(t) - \int_0^\infty \theta'(t-\tau)\overline{x}(\tau)\,\mathrm{d}\tau \ =\ \int_0^\infty \theta'(t-\tau)\Big(\overline{x}(t) - \overline{x}(\tau) - s_*(t-\tau)\Big)\,\mathrm{d}\tau \ ,$$

and the right-hand side converges to zero as $t \to +\infty$ by Lemma 7.4. Thus, if $n \in \mathbb{N}$ is sufficiently large, we see that

$$\overline{x}(t) - n - 2 \ \le\ x_+(t) \ \le\ \overline{x}(t) - n + 1 \ , \quad \text{for } t \in [t_n, t_{n+1}] \ . \tag{8.3}$$

Similarly, since $\int_{\mathbb{R}} \theta''(x)\,\mathrm{d}x = 0$ and $\int_{\mathbb{R}} x\theta''(x)\,\mathrm{d}x = -1$, we have

$$s_* - \int_0^\infty \theta''(t-\tau)\overline{x}(\tau)\,\mathrm{d}\tau \;=\; \int_0^\infty \theta''(t-\tau)\Big(\overline{x}(t) - \overline{x}(\tau) - s_*(t-\tau)\Big)\,\mathrm{d}\tau \;\xrightarrow[t\to+\infty]{}\; 0\;,$$

hence $|x_+'(t) - s_*| \le 1$ if $t \ge 0$ is sufficiently large.

On the other hand, using the assumption (1.9) on the initial data and proceeding exactly as in the proof of Proposition 3.2, we see that there exists $\xi_1 \in \mathbb{R}$ such that

$$\sup_{z\le \xi_1 - t/\sqrt{\alpha}} \int_z^{z+1} \Big(|\dot u(x,t)|^2 + |u'(x,t)|^2 + |u(x,t)-1|^2\Big)\,\mathrm{d}x \;\le\; K_1\delta_1\, e^{-\mu_1 t}\;. \tag{8.4}$$

For all $t \ge 0$, we set $x_-(t) = \xi_1 - 2t/\sqrt{\alpha}$. Without loss of generality, we can assume that $x_-(t) \le x_+(t)$ for all $t \ge t_0$.

We next define, for all $t \ge t_0$,

$$\Phi(t) \;=\; \int_{\mathbb{R}} \phi(x,t)\Big(\frac{\alpha}{2}|\dot u(x,t)|^2 + \frac{1}{2}|u'(x,t)|^2 + \overline{V}(u(x,t))\Big)\,\mathrm{d}x\;,$$

where $\overline{V}(u) = V(u) - V(1) \ge 0$ and

$$\phi(x,t) \;=\; \begin{cases} e^{x-x_-(t)} & \text{if} \;\; x \le x_-(t) \\ 1 & \text{if} \;\; x_-(t) \le x \le x_+(t) \\ e^{x_+(t)-x} & \text{if} \;\; x \ge x_+(t) \end{cases}\;.$$

A direct calculation shows that

$$\begin{aligned}\Phi'(t) \;=\; &-\int_{\mathbb{R}} \phi(x,t)|\dot u(x,t)|^2\,\mathrm{d}x - \int_{-\infty}^{x_-(t)} \phi\Big\{x_-'(t)\Big(\frac{\alpha}{2}|\dot u|^2 + \frac{1}{2}|u'|^2 + \overline{V}(u)\Big) + \dot u u'\Big\}\,\mathrm{d}x \\ &+ \int_{x_+(t)}^{\infty} \phi\Big\{x_+'(t)\Big(\frac{\alpha}{2}|\dot u|^2 + \frac{1}{2}|u'|^2 + \overline{V}(u)\Big) + \dot u u'\Big\}\,\mathrm{d}x\;.\end{aligned}$$

As is clear from (8.4), the second integral in the right-hand side converges to zero as $t \to +\infty$. Since $x_+(t) - \overline{x}(t) \to -\infty$ by (8.3), this is also the case for the last integral. Indeed, as $v_*(x) \to 1$ when $x \to -\infty$, it follows from (8.2), (8.3) that $(u,\dot u)(x_+(t)+\cdot, t)$ converges to $(1,0)$ in $H^1_{\mathrm{loc}}(\mathbb{R}) \times L^2_{\mathrm{loc}}(\mathbb{R})$ as $t \to +\infty$. Since $\Phi(t) \ge 0$ for all times, we conclude that, given any $T > 0$, there exists a sequence $t_n' \to +\infty$ such that

$$\int_{t_n'-T}^{t_n'+T} \int_{\mathbb{R}} \phi(x,t)|\dot u(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t \;\xrightarrow[n\to\infty]{}\; 0\;. \tag{8.5}$$

Now, let us denote by $\Psi(z,t_n')$ the quantity

$$\Psi(z,t_n') = \int_z^{z+1} \Big(|\dot u(x,t_n')|^2 + |u'(x,t_n')|^2 + |u(x,t_n')-1|^2\Big)\,\mathrm{d}x\;.$$

We claim that

$$\sup_{z\in[x_-(t_n'),x_+(t_n')]} \Psi(z,t_n') \;\xrightarrow[n\to\infty]{}\; 0\;. \tag{8.6}$$

Indeed, we first observe that, due to (8.2), (8.3), and (8.4), one has

$$\sup\left\{\Psi(z,t_n')\,\middle|\, z\in\left[x_-(t_n'),x_-(t_n')+\frac{t_n'}{\sqrt{\alpha}}\right]\;\cup\;\left[x_+(t_n')-n+2,x_+(t_n')\right]\right\}\;\xrightarrow[n\to\infty]{}\;0\,. \quad (8.7)$$

Assume that (8.6) does not hold, so that after extracting a subsequence the left-hand side of (8.6) is bounded from below for all $n\in\mathbb{N}$ by some $\varepsilon>0$. Then, using (8.7) and a continuity argument, we can find for any $\varepsilon'\in(0,\varepsilon)$ a sequence $\{z_n\}$ such that $z_n\in[x_-(t_n')+t_n'/\sqrt{\alpha},x_+(t_n')-n+2]$ for all $n\in\mathbb{N}$ and $\Psi(z_n,t_n')=\varepsilon'$. Without loss of generality, we can assume that $\varepsilon'>0$ is sufficiently small so that the following property holds: if $w:\mathbb{R}\to\mathbb{R}$ is a bounded solution of the differential equation $w''-V'(w)=0$ such that $|w(0)-1|^2\le 2\varepsilon'$, then $w(x)=1$ for all $x\in\mathbb{R}$. The existence of such an ε' follows from our assumptions (1.3)–(1.6) on the potential V.

Using once again Proposition 2.3, we can assume that, after extracting a subsequence, the sequence of functions $(u,\dot u)(z_n+\cdot,t_n'+\cdot)$ converges in the space $\mathcal{C}^0([-T,T],H^1_{\mathrm{loc}}(\mathbb{R})\times L^2_{\mathrm{loc}}(\mathbb{R}))$ to some limit $(w,\dot w)$ which satisfies (1.1). In view of (8.5), we have $\dot w\equiv 0$, so that $w:\mathbb{R}\to\mathbb{R}$ is a bounded solution of the differential equation $w''-V'(w)=0$. Moreover,

$$|w(0)-1|^2\;\le\;2\int_0^1\left(|w'(x)|^2+|w(x)-1|^2\right)\mathrm{d}x\;=\;2\varepsilon'\,,$$

hence our assumption on ε' implies that $w\equiv 1$, which is clearly absurd. Thus (8.6) is established, and using in addition (8.2), (8.4) we see that (8.1) holds for the sequence $\{t_n'\}$. This concludes the proof. □

9 Local stability of the travelling front

The aim of this final section is to show that the family (1.8) of travelling fronts of (1.1) is *asymptotically stable with shift* in the uniformly local energy space $X=H^1_{\mathrm{ul}}(\mathbb{R})\times L^2_{\mathrm{ul}}(\mathbb{R})$. Together with Proposition 8.1, this will conclude the proof of Theorem 1.1. Whereas a lot is known about local stability of travelling fronts in parabolic systems (see e.g. [42]), for the hyperbolic equation (1.1) with a bistable potential we are only aware of the note [10] where local stability in the usual energy space $H^1(\mathbb{R})\times L^2(\mathbb{R})$ is briefly discussed.

From now on, we fix $c=c_*$ and we denote by $h:\mathbb{R}\to(0,1)$ the unique solution of (1.7) such that $h(0)=\varepsilon_0$. Linearizing (1.13) at the steady state $u_c=h$, we obtain the evolution equation

$$\alpha u_{tt}+u_t-2\alpha c u_{yt}\;=\;u_{yy}+cu_y-g(y)u\,, \quad (9.1)$$

where $g(y)=V''(h(y))$. Proceeding as in Section 2, it is straightforward to verify that (9.1) defines a C_0-group $\{S(t)\}_{t\in\mathbb{R}}$ of bounded linear operators in X, the generator of which is the linear operator $\mathcal{A}$ given by

$$D(\mathcal{A})\;=\;Y\,,\qquad \mathcal{A}\;=\;\frac{1}{\alpha}\begin{pmatrix}0 & \alpha\\ \partial_y^2+c\partial_y-g(y) & -1+2\alpha c\partial_y\end{pmatrix}\,, \quad (9.2)$$

where $Y = H^2_{\rm ul}(\mathbb{R}) \times H^1_{\rm ul}(\mathbb{R})$. By translation invariance, $\lambda = 0$ is an eigenvalue of $\mathcal{A}$ with eigenfunction $(h', 0)$. This eigenvalue is in fact simple, and the corresponding spectral projection reads

$$\Pi \begin{pmatrix} u \\ v \end{pmatrix} \;=\; N \begin{pmatrix} h' \\ 0 \end{pmatrix} \int_{\mathbb{R}} (\psi_1 u + \psi_2 v)\,\mathrm{d}y \;, \tag{9.3}$$

where $\psi_2(y) = e^{cy} h'(y)$, $\psi_1 = \alpha^{-1}\psi_2 + 2c\psi_2'$, and $N > 0$ is a normalization factor. One can check that ψ_1, ψ_2 decay exponentially to zero as $|y| \to \infty$. The main result of this section is:

Proposition 9.1. *There exist positive constants C_0 and ν such that*

$$\|S(t)(\mathbf{1} - \Pi)\|_{\mathcal{L}(X)} \;\le\; C_0\, e^{-\nu t} \;, \quad \textit{for all } t \ge 0 \;. \tag{9.4}$$

Using Proposition 9.1 and classical arguments which can be found in [42] or [22], it is easy to show that the family of all translates of the steady state $(h, 0)$ is *normally hyperbolic* and *asymptotically stable* for the evolution defined by (1.13) on X. In other words, any solution of (1.13) whose initial data lie in a small tubular neighborhood of this family of equilibria (in the topology of X) converges exponentially fast as $t \to +\infty$ to some element of the family. Now, if $u(x,t)$ is a solution of (1.1) satisfying the assumptions of Theorem 1.1, Proposition 8.1 shows that the corresponding solution of (1.13) eventually enters such a tubular neighborhood. Thus there exists $x_0 \in \mathbb{R}$ such that

$$\|u_c(\cdot,t) - h(\cdot - x_0)\|_{H^1_{\rm ul}} + \|\dot u_c(\cdot,t)\|_{L^2_{\rm ul}} \;=\; \mathcal{O}(e^{-\nu t}) \;, \quad \text{as } t \to +\infty \;, \tag{9.5}$$

which implies (1.11) and concludes the proof of Theorem 1.1.

It remains to prove Proposition 9.1. Let

$$g_\infty(y) \;=\; V''(1)\,\frac{1 - \tanh(y)}{2} + V''(0)\,\frac{1 + \tanh(y)}{2} \;, \quad y \in \mathbb{R} \;.$$

Obviously $g_\infty(y) \ge m$ for all $y \in \mathbb{R}$, where $m = \min(V''(0), V''(1)) > 0$. Let $\mathcal{A}_\infty$ be the linear operator on X obtained by replacing g with g_∞ in the definition (9.2) of $\mathcal{A}$, and let $S_\infty(t)$ be the C_0-group generated by $\mathcal{A}_\infty$. For any $u \in H^1_{\rm ul}(\mathbb{R})$, the map $y \mapsto (g(y) - g_\infty(y))u(y)$ belongs to $H^1(\mathbb{R})$ and converges exponentially to zero as $y \to \pm\infty$. In particular, the linear operator $\mathcal{A} - \mathcal{A}_\infty : \; (u,v) \mapsto \alpha^{-1}(0, (g_\infty - g)u)$ is *compact* in X. Moreover, the group $S_\infty(t)$ is bounded in $H^2(\mathbb{R}) \times H^1(\mathbb{R})$ and, due to the finite speed of propagation, it preserves the exponential decay in space. Therefore, the Duhamel perturbation formula

$$S(t)w \;=\; S_\infty(t)w + \int_0^t S_\infty(t-\tau)(\mathcal{A} - \mathcal{A}_\infty)S(\tau)w\,\mathrm{d}\tau \;, \quad w \in X \;,$$

shows that $S(t) - S_\infty(t)$ is compact in X for any $t \in \mathbb{R}$. In particular, $S(t)$ and $S_\infty(t)$ have the same essential spectrum (in what follows, we use the notion of essential spectrum adopted in [22]). The first step in the proof of Proposition 9.1 is:

Lemma 9.2. *There exist positive constants C_1 and ν_∞ such that*

$$\|S_\infty(t)\|_{\mathcal{L}(X)} \;\le\; C_1\, e^{-\nu_\infty t} \;, \quad \textit{for all } t \ge 0 \;.$$

Proof: We proceed as in the proof of Proposition 2.1. Let $\rho(y) = \exp(-\kappa|y|)$, where $\kappa > 0$ is small enough so that $\kappa^2 + c\kappa \le \min(m/2, (8\alpha)^{-1})$ and $16\alpha\kappa^2c^2 \le m$. If $u(x,t)$ is a solution of the evolution equation $\alpha u_{tt} + u_t - 2\alpha c u_{yt} = u_{yy} + cu_y - g_\infty(y)u$ associated with $S_\infty(t)$, we define for all $\xi \in \mathbb{R}$ and all $t \ge 0$:

$$\hat{\mathcal{E}}(\xi,t) \;=\; \int_{\mathbb{R}} \rho(y-\xi)\Big(\alpha^2 u_t^2 + \alpha u_y^2 + \alpha g_\infty u^2 + \frac{1}{2}u^2 + \alpha u u_t\Big)(y,t)\,\mathrm{d}y\ .$$

Since $|\alpha u u_t| \le \frac{1}{2}(u^2 + \alpha^2 u_t^2)$, there exists a constant $C_1 \ge 1$ such that

$$C_1^{-1}\|(u(\cdot,t),\dot u(\cdot,t))\|_X^2 \;\le\; \sup_{\xi\in\mathbb{R}} \hat{\mathcal{E}}(\xi,t) \;\le\; C_1\|(u(\cdot,t),\dot u(\cdot,t))\|_X^2\ . \tag{9.6}$$

On the other hand, differentiating $\hat{\mathcal{E}}(\xi,t)$ with respect to t and using our assumptions on κ, we obtain

$$\begin{aligned}
\partial_t\hat{\mathcal{E}}(\xi,t) \;&=\; -\int_{\mathbb{R}} \rho(y-\xi)(\alpha u_t^2 + u_y^2 + g_\infty u^2)(y,t)\,\mathrm{d}y \\
&\quad - \int_{\mathbb{R}} \rho'(y-\xi)\Big(2\alpha u_y u_t + 2\alpha^2 c u_t^2 + 2\alpha c u u_t + u u_y + \frac{c}{2}u^2\Big)(y,t)\,\mathrm{d}y \\
&\le -\frac{1}{2}\int_{\mathbb{R}} \rho(y-\xi)(\alpha u_t^2 + u_y^2 + g_\infty u^2)(y,t)\,\mathrm{d}y \;\le\; -2\nu_\infty \hat{\mathcal{E}}(\xi,t)\ ,
\end{aligned}$$

for some $\nu_\infty > 0$. Thus $\hat{\mathcal{E}}(\xi,t) \le \hat{\mathcal{E}}(\xi,0)\,e^{-2\nu_\infty t}$ for all $t \ge 0$, and using in addition (9.6) we obtain the desired estimate. □

Since $S(t)$ is a compact perturbation of $S_\infty(t)$ for any $t \ge 0$, it follows from Lemma 9.2 that the spectrum of $S(t)$ outside the disk $\{z \in \mathbb{C}\,|\,|z| \le e^{-\nu_\infty t}\}$ consists of isolated eigenvalues with finite multiplicities. By the spectral mapping theorem, any such eigenvalue has the form $e^{\lambda t}$, where $\lambda \in \mathbb{C}$ is an eigenvalue of the generator $\mathcal{A}$. If $w = (u,v) \in Y$ satisfies $\mathcal{A}w = \lambda w$, it follows from (9.2) that $v = \lambda u$ and

$$u'' + c(1+2\alpha\lambda)u' - g(y)u \;=\; \lambda(1+\alpha\lambda)u\ . \tag{9.7}$$

It remains to study the nonlinear eigenvalue problem (9.7). Let

$$\mu_\alpha \;=\; \frac{1}{2\alpha}\Big(-1 + \mathrm{Re}\sqrt{1-4\alpha m}\Big) \;<\; 0\ ,$$

where $m = \min(V''(0), V''(1)) > 0$. One can check that $\mu_\alpha = \sup\{\mathrm{Re}(\lambda)\,|\,\lambda \in \sigma_{\mathrm{ess}}(\mathcal{A})\}$. The key observation is:

Proposition 9.3. *The spectrum of $\mathcal{A}$ in the half-plane $\{z \in \mathbb{C}\,|\,\mathrm{Re}(z) > \mu_\alpha\}$ consists of a finite sequence of simple real eigenvalues $0 = \lambda_0 > \lambda_1 > \cdots > \lambda_k > \mu_\alpha$, where $k \in \mathbb{N}$ depends on α and V.*

The proof of Proposition 9.3 relies on the following elementary result:

Lemma 9.4. *Assume that $z \in \mathbb{C}$ satisfies $\mathrm{Re}(z) > \mu_\alpha$, and let*

$$\gamma \,=\, \frac{c}{2}(1 + 2\alpha z)\;, \quad \delta \,=\, m + z(1+\alpha z)\;.$$

Then $\mathrm{Re}\sqrt{\gamma^2+\delta} > \mathrm{Re}(\gamma) > 0$.

Proof: Since $\mathrm{Re}(z) > \mu_\alpha \ge -1/(2\alpha)$, it is clear that $\mathrm{Re}(\gamma) > 0$. On the other hand, if $z > 0$, we have $\gamma > 0$ and $\delta > 0$, hence $\sqrt{\gamma^2+\delta} > \gamma$. Thus, to prove Lemma 9.4, it is sufficient to verify that the equality $\mathrm{Re}\sqrt{\gamma^2+\delta} = \mathrm{Re}(\gamma)$ cannot occur if $\mathrm{Re}(z) > \mu_\alpha$. Assume on the contrary that $\sqrt{\gamma^2+\delta} = \gamma + i\beta$ for some $\beta \in \mathbb{R}$. Then $\delta = 2i\gamma\beta - \beta^2$, and if we set $z = z_1 + iz_2$ with $z_1, z_2 \in \mathbb{R}$ we obtain the relation

$$m + z_1 + \alpha(z_1^2 - z_2^2) + iz_2(1+2\alpha z_1) \,=\, -2\alpha\beta c z_2 - \beta^2 + i\beta c(1+2\alpha z_1)\;. \tag{9.8}$$

Taking the imaginary part of both sides, we find $z_2 = \beta c$, because $1 + 2\alpha z_1 > 0$ by assumption. Using this information and taking the real part of (9.8), we arrive at

$$\alpha\Bigl(z_1 + \frac{1}{2\alpha}\Bigr)^2 + \Bigl(\alpha + \frac{1}{c^2}\Bigr)^2 z_2^2 \,=\, \frac{1}{4\alpha} - m\;. \tag{9.9}$$

This is clearly impossible if $4\alpha m > 1$. In the converse case, equation (9.9) defines an ellipse in $\mathbb{C}$ which is entirely contained in the half-plane $\{\mathrm{Re}(z) \le \mu_\alpha\}$, thus contradicting our assumption on z. □

Proof of Proposition 9.3: Assume that $\lambda \in \sigma(\mathcal{A})$ satisfies $\mathrm{Re}(\lambda) > \mu_\alpha$. Then λ is an eigenvalue of $\mathcal{A}$, and there exists a nonzero $u \in H^2_{\mathrm{ul}}(\mathbb{R})$ satisfying (9.7). Since $g(y)$ converges exponentially to $V''(0)$ as $y \to +\infty$, we know that $u(y) = A\phi_1(y) + B\phi_2(y)$ for some $A, B \in \mathbb{C}$, where ϕ_1, ϕ_2 are particular solutions of (9.7) satisfying

$$\lim_{y\to+\infty} \phi_1(y)\, e^{\gamma y} e^{\sqrt{\gamma^2+\delta_+}\, y} \,=\, 1\;, \quad \lim_{y\to+\infty} \phi_2(y)\, e^{\gamma y} e^{-\sqrt{\gamma^2+\delta_+}\, y} \,=\, 1\;,$$

where $\gamma = \frac{c}{2}(1+2\alpha\lambda)$ and $\delta_+ = V''(0) + \lambda(1+\alpha\lambda)$, see [3, Section 3.8]. But Lemma 9.4 implies that $\mathrm{Re}\sqrt{\gamma^2+\delta_+} > \mathrm{Re}(\gamma) > 0$, hence we must have $B = 0$ because $\phi_2(y)$ is unbounded as $y \to +\infty$. Thus

$$u(y) \,=\, A\phi_1(y) \,\approx\, A\, e^{-\gamma y}\, e^{-\sqrt{\gamma^2+\delta_+}\, y}\;, \quad \text{as } y \to +\infty\;,$$

and a similar argument shows that

$$u(y) \,\approx\, C\, e^{-\gamma y}\, e^{\sqrt{\gamma^2+\delta_-}\, y}\;, \quad \text{as } y \to -\infty\;,$$

for some $C \in \mathbb{C}$, where $\delta_- = V''(1) + \lambda(1+\alpha\lambda)$. These observations reveal in particular that the bounded solutions of (9.7) form a one-dimensional family, hence λ is a simple eigenvalue of $\mathcal{A}$.

Moreover, if we set $U(y) = e^{\gamma y}u(y)$, then $U(y)$ decays exponentially to zero as $y \to \pm\infty$, and a direct calculation shows that U solves the differential equation

$$U'' - \Bigl(g(y) + \frac{c^2}{4}\Bigr)U \,=\, \lambda(1+\alpha\lambda)(1+\alpha c^2)U\;, \quad y \in \mathbb{R}\;. \tag{9.10}$$

Thus $\mu = \lambda(1+\alpha\lambda)(1+\alpha c^2)$ is an eigenvalue of the selfadjoint operator $\mathcal{L} = \partial_y^2 - (g+c^2/4)$. In particular, we have $\mu \in \mathbb{R}$, hence $\lambda \in \mathbb{R}$ because $\mathrm{Re}(\lambda) > -1/(2\alpha)$. Furthermore, since $\mu = 0$ is an eigenvalue of $\mathcal{L}$ with eigenfunction $U(y) = e^{\gamma y}h'(y) < 0$, we know from Sturm-Liouville theory that all the other eigenvalues of $\mathcal{L}$ are strictly negative. Finally, it follows from Bargmann's bound and the min-max principle that $\mathcal{L}$ has only a finite number of eigenvalues, see e.g. [43]. We conclude that the spectrum of $\mathcal{A}$ in the half-plane $\{z \in \mathbb{C} \,|\, \mathrm{Re}(z) > \mu_\alpha\}$ consists of the eigenvalue $\lambda = 0$ and, possibly, of a finite number of negative eigenvalues. □

It is now easy to conclude the proof of Proposition 9.1. We know from Proposition 9.3 that $\lambda = 0$ is a simple, isolated eigenvalue of $\mathcal{A}$, and that the rest of the spectrum lies in the half-plane $\{\mathrm{Re}(\lambda) \le -\hat{\nu}\}$ for some $\hat{\nu} > 0$. Going back to the semigroup $S(t)$ generated by $\mathcal{A}$, we infer that for any $t > 0$ the spectrum of $S(t)$ is entirely contained in the disk $\{z \in \mathbb{C} \,|\, |z| \le e^{-\nu t}\}$, where $\nu = \min(\nu_\infty, \hat{\nu})$ and ν_∞ is as in Lemma 9.2, except for the simple eigenvalue $z = 1$ which is due to the translation invariance. If we remove that eigenvalue by restricting $S(t)$ to the invariant subspace $\ker \Pi$, where Π is the spectral projection (9.3), we obtain estimate (9.4), possibly with a slightly smaller ν. The proof of Theorem 1.1 is now complete.

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