Identification of viscosity in an incompressible fluid

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Non-Newtonian fluids: viscosity depends on the shear rate

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Non-Newtonian fluids: viscosity depends on the shear rate

We are interested in determining the viscosity function by a non-invasive method.

Navier-Stokes Equations

$$\partial_t u + u \cdot \nabla u = \mathrm{div}(\mu \mathsf{Sym}(\nabla u) - pI) \quad \text{in} \quad \mathbb{R}^3 \tag{1}$$

and

$$\mathrm{div} u = 0 \quad \text{in} \quad \mathbb{R}^3, \tag{2}$$

where $u = (u^1, u^2, u^3)^T$ is the velocity and p is the pressure,

$$\mathsf{Sym}(\nabla u) = \frac{1}{2}(\nabla u + \nabla u^T).$$

$\mu > 0$: viscosity function

Eq (2): incompressibility condition

When μ is a positive constant, (1) and (2) are reduced to the following familiar form:

$$\begin{cases} \partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla p = 0, \\ \mathrm{div} u = 0. \end{cases}$$

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$$\begin{cases} \partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla p = 0, \\ \operatorname{div} u = 0. \end{cases}$$

If u is time-independent, then (1), (2) become:

$$\begin{cases} \operatorname{div}(\mu \mathsf{Sym}(\nabla u)) - \nabla p - u \cdot \nabla u = 0, \\ \operatorname{div} u = 0. \end{cases} \tag{4}$$

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This problem is *nonlinear*^{2} (nonlinear inverse problem for nonlinear equation)

Mathematical setup

Ω: open and bounded in $\mathbb{R}^3$ with $\partial\Omega \in C^\infty$

Consider the boundary value problem:

$$\begin{cases} \mathrm{div}(\mu \mathsf{Sym}(\nabla u) - pI) - u \cdot \nabla u = 0 \quad \text{in} \quad \Omega, \\ \mathrm{div} u = 0 \quad \text{in} \quad \Omega, \\ u = f \quad \text{on} \quad \partial\Omega, \end{cases} \tag{5}$$

where f is appropriately chosen and satisfies

$$\int_{\partial\Omega} f \cdot \mathbf{n} ds = 0, \tag{6}$$

where $\mathbf{n}$ is the unit outer normal of $\partial\Omega$.

Assume that a solution to (5) exists and the trace

$$\sigma_\mu(u,p)\mathbf{n}|_{\partial\Omega}$$

is well defined, where

$$\sigma_\mu(u,p) := \mu\mathsf{Sym}(\nabla u) - pI$$

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Physically, $\sigma_\mu(u,p)\mathbf{n}|_{\partial\Omega}$ is the Cauchy forces acting on the boundary $\partial\Omega$.

Define the set of Cauchy data for (5):

$$\widetilde{S}_\mu = \{(u|_{\partial\Omega}, \sigma_\mu \mathbf{n}|_{\partial\Omega}) : u \text{ solves } (5)\}$$

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IP: determine μ from $\widetilde{S}_\mu$

First study the uniqueness question:

$$\widetilde{S}_{\mu_1} = \widetilde{S}_{\mu_2} \Rightarrow \mu_1 = \mu_2\ ?$$

Theorem 1 [X. Li and W.] *Assume that* $\mu_1(x),\ \mu_2(x) \in C^{n_0}(\overline{\Omega})$ *for* $n_0 \geq 8$ *and*

$$\partial^\alpha \mu_1(x) = \partial^\alpha \mu_2(x) \quad \forall\, x \in \partial\Omega,\ |\alpha| \leq 1. \tag{7}$$

If $\widetilde{S}_{\mu_1} = \widetilde{S}_{\mu_2}$ *then* $\mu_1 = \mu_2$*.*

We can remove assumption (7) for some domains Ω.

Theorem 2 [Boundary determination] *Let* $\partial\Omega$ *be convex with nonvanishing Gauss curvature. If* $\widetilde{S}_{\mu_1} = \widetilde{S}_{\mu_2}$ *then* $\mu_1(x) = \mu_2(x)$ *and* $\nabla\mu_1(x) \cdot \mathbf{n} = \nabla\mu_2(x) \cdot \mathbf{n}$ *for all* $x \in \partial\Omega$*.*

Putting together Theorem 1 and 2 gives

Theorem 3 *Let $\partial\Omega$ be convex with nonvanishing Gauss curvature. Assume that $\mu_1(x)$ and $\mu_2(x)$ are two viscosity functions satisfying $\mu_1, \mu_2 \in C^{n_0}(\overline{\Omega})$ for $n_0 \geq 8$. If $\widetilde{S}_{\mu_1} = \widetilde{S}_{\mu_2}$ then $\mu_1 = \mu_2$.*

The restriction on the geometry of Ω is due to the compatibility condition (6). This is a global restriction on the Dirichlet data.

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Taking $\varepsilon \rightarrow 0$, we expect that $v_\varepsilon \rightarrow v_0$, $q_\varepsilon \rightarrow q_0$ and (v_0, q_0) satisfies

$$\begin{cases} \mathrm{div}(\mu \mathsf{Sym}(\nabla v_0)) - \nabla q_0 = 0, \\ \mathrm{div} v_0 = 0. \end{cases} \tag{11}$$

Eq.(8) is the Stokes system (linear). The Stokes system has its own Cauchy data on the boundary defined by

$$S_\mu = \{(v_0|_{\partial\Omega}, \sigma_\mu(v_0, q_0)\mathbf{n}|_{\partial\Omega}) : (v_0, q_0) \text{ solves (8)}\}.$$

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Question: $\widetilde{S}_{\mu_1} = \widetilde{S}_{\mu_2} \Rightarrow S_{\mu_1} = S_{\mu_2}$?

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Consider the boundary value problem:

$$\begin{cases} \mathrm{div}(\mu \mathsf{Sym}(\nabla u_\varepsilon) - p_\varepsilon I) - u_\varepsilon \cdot \nabla u_\varepsilon = 0 \quad \text{in} \quad \Omega, \\ \mathrm{div} u_\varepsilon = 0 \quad \text{in} \quad \Omega, \\ u_\varepsilon = \varepsilon\phi \quad \text{on} \quad \partial\Omega \end{cases} \tag{14}$$

with ϕ satisfying the compatibility condition (6).

Let $(u_\varepsilon^{(j)}, p_\varepsilon^{(j)})$ be solutions of (12) corresponding to μ_j, $j = 1, 2$. If $\widetilde{S}_{\mu_1} = \widetilde{S}_{\mu_2}$, then

$$u_\varepsilon^{(1)}|_{\partial\Omega} = u_\varepsilon^{(2)}|_{\partial\Omega}$$

and

$$\sigma_{\mu_1}(u_\varepsilon^{(1)}, p_\varepsilon^{(1)})\mathbf{n}|_{\partial\Omega} = \sigma_{\mu_2}(u_\varepsilon^{(2)}, p_\varepsilon^{(2)})\mathbf{n}|_{\partial\Omega}.$$

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Dividing by ε:

$$\varepsilon^{-1}u_\varepsilon^{(1)}|_{\partial\Omega} = \varepsilon^{-1}u_\varepsilon^{(2)}|_{\partial\Omega}$$

and

$$\varepsilon^{-1}\sigma_{\mu_1}(u_\varepsilon^{(1)}, p_\varepsilon^{(1)})\mathbf{n}|_{\partial\Omega} = \varepsilon^{-1}\sigma_{\mu_2}(u_\varepsilon^{(2)}, p_\varepsilon^{(2)})\mathbf{n}|_{\partial\Omega}.$$

Let $\varepsilon \to 0$ then we get that

$$v_0^{(1)}|_{\partial\Omega} = v_0^{(2)}|_{\partial\Omega}$$

and

$$\sigma_{\mu_1}(v_0^{(1)}, q_0^{(1)})\mathbf{n}|_{\partial\Omega} = \sigma_{\mu_2}(v_0^{(2)}, q_0^{(2)})\mathbf{n}|_{\partial\Omega},$$

where $(v_0^{(j)}, q_0^{(j)})$ are solutions of Stokes system (8) corresponding to μ_j, $j = 1, 2$.

In other words, $S_{\mu_1} = S_{\mu_2}$.

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Rigorously, we set $u_\varepsilon = \varepsilon v_0 + v_\varepsilon$ and $p_\varepsilon = \varepsilon q_0 + q_\varepsilon$, where (v_0, q_0) is a solution to the Stokes system. The task is to show that $(v_\varepsilon, q_\varepsilon)$ exists and

$$\frac{v_\varepsilon}{\varepsilon} \to 0, \quad \frac{p_\varepsilon}{\varepsilon} \to 0$$

as $\varepsilon \to 0$ in some nice space.

Inverse problem for the Stokes system

Now the inverse problem for the Navier-Stokes equations can be solved if we can solve the same problem for the Stokes equations, i.e., we want to prove

$$S_{\mu_1} = S_{\mu_2} \Rightarrow \mu_1 = \mu_2.$$

This is indeed true.

Theorem 4 [Heck-Li-W] *Assume that* $\mu_1(x),\ \mu_2(x) \in C^{n_0}(\overline{\Omega})$ *for* $n_0 \geq 8$ *and*

$$\partial^\alpha \mu_1(x) = \partial^\alpha \mu_2(x) \quad \forall\, x \in \partial\Omega,\ |\alpha| \leq 1. \tag{15}$$

If $S_{\mu_1} = S_{\mu_2}$ *then* $\mu_1 = \mu_2$*.*

Remark 5 *As before, the boundary restriction* (15) *can be removed if* $\partial\Omega$ *has nonvanishing Gauss curvature.*

Ideas in proving Theorem 4

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Green's formula:

$$S_{\mu_1} = S_{\mu_2} \Rightarrow$$

$$\int_\Omega (\mu_1 - \mu_2)\text{Sym}(\nabla u_1) \cdot \overline{\text{Sym}(\nabla u_2)} dx = 0, \tag{17}$$

where (u_j, p_j) ($j = 1, 2$) is the solution of

$$\begin{cases} \text{div}(\mu_j \text{Sym}(\nabla u_j)) - \nabla p_j = 0 \quad \text{in} \quad \Omega, \\ \text{div} u_j = 0 \quad \text{in} \quad \Omega. \end{cases}$$

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So the strategy now is to find enough u_1 and u_2 plugging into the identity such that we can conclude $\mu_1 - \mu_2 = 0$.

We will look for special u of the form

$$u = e^{ix\cdot\zeta} u_r,$$

where $\zeta \in \mathbb{C}^3$ satisfying $\zeta \cdot \zeta = 0$ and we have a good control of u_r in terms of the size of ζ (so-called complex geometrical optics solutions).

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This is no easy task because the Stokes system is strongly coupled in the leading order and two equations are in different orders.

A New System

Stokes equation for (u, p) $- - - - - - \rightarrow$ A new system for (w, f)

$\uparrow$

relationship between

(u, p) and (w, f)

$\downarrow$

Special solutions of the Stokes equation $\leftarrow - - - - - -$ Special solutions o the new system

Setting

$$u = \mu^{-1/2} w + \mu^{-1} \nabla f - f \nabla \mu^{-1}$$

Let $\begin{pmatrix} w \\ f \end{pmatrix}$ solve

$$\Delta \begin{pmatrix} w \\ f \end{pmatrix} + A_1(x) \begin{pmatrix} \nabla f \\ \mathrm{div} w \end{pmatrix} + A_0(x) \begin{pmatrix} w \\ f \end{pmatrix} = 0$$

with

$$A_1(x) = \begin{pmatrix} -2\mu^{1/2}\nabla^2\mu^{-1} & -\mu^{-1}\nabla\mu \\ 0 & \mu^{1/2} \end{pmatrix}.$$

The form of A_0 is not important.

Then

$$u = \mu^{-1/2} w + \mu^{-1} \nabla f - f \nabla \mu^{-1}$$

and

$$p = \mathrm{div}(\mu^{1/2} w) + 2\Delta f$$

solve the Stokes system.

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solve the Stokes system.

The idea of deriving the new system is motivated by a similar reduction for the elasticity. One key observation is that the identity does not contain the pressure p.

Not so technical part

Choice of ζ: Let l, α and β be pairwise orthogonal vectors in $\mathbb{R}^3$ with $|\alpha| = |\beta| = 1$. Let $\theta = \alpha + i\beta$. For $\tau \gg 0$, we set

$$\zeta_1(\tau) = \frac{l}{2} + \sqrt{\tau^2 - \frac{|l|^2}{4}}\,\alpha + i\tau\beta$$

$$\zeta_2(\tau) = -\frac{l}{2} + \sqrt{\tau^2 - \frac{|l|^2}{4}}\,\alpha - i\tau\beta$$

Construction of special solutions

$$\begin{pmatrix} w_1 \\ f_1 \end{pmatrix} = e^{ix\cdot\zeta_1} \begin{pmatrix} r_1 \\ s_1 \end{pmatrix}, \qquad \begin{pmatrix} r_1 \\ s_1 \end{pmatrix} = C_1(x,\theta) g_1(\theta\cdot x) + O(\tau^{-1})$$

where C_1 is a solution of

$$-2\theta\cdot\nabla C_1(x,\theta) = \begin{pmatrix} -2\mu_1^{1/2}\nabla^2\mu_1^{-1} & -\mu_1^{-1}\nabla\mu_1 \\ 0 & \mu_1^{1/2} \end{pmatrix} \begin{pmatrix} 0_{3\times 3} & \theta \\ \theta^T & 0 \end{pmatrix} C_1(x,\theta)$$

and $g_1(z)$ is an arbitrary polynomial of z.

$$\begin{pmatrix} w_2 \\ f_2 \end{pmatrix} = e^{ix\cdot\zeta_2} \begin{pmatrix} r_2 \\ s_2 \end{pmatrix}, \quad \begin{pmatrix} r_2 \\ s_2 \end{pmatrix} = C_2(x,\bar{\theta}) g_2(\bar{\theta}\cdot x) + O(\tau^{-1})$$

where C_2 is a solution of

$$-2\bar{\theta}\cdot\nabla C_2(x,\bar{\theta}) = \begin{pmatrix} -2\mu_2^{1/2}\nabla^2\mu_2^{-1} & -\mu_2^{-1}\nabla\mu_2 \\ 0 & \mu_2^{1/2} \end{pmatrix} \begin{pmatrix} 0_{3\times 3} & \bar{\theta} \\ \bar{\theta}^T & 0 \end{pmatrix} C_2(x,\bar{\theta})$$

and $g_2(\bar{z})$ is an arbitrary polynomial of $\bar{z}$.

Define

$$u_j = \mu_j^{-1/2} w_j + \mu_j^{-1} \nabla f_j - f \nabla \mu_j^{-1}$$

Denote

$$H(u_1, u_2) = \int_\Omega (\mu_1 - \mu_2) \text{Sym}(\nabla u_1) \cdot \overline{\text{Sym}(\nabla u_2)} dx$$

Then the leading term in the express $H(u_1, u_2)$ is of order 2 in τ. We will prove that μ can be determined uniquely from

$$\lim_{\tau \to \infty} \tau^{-2} H(u_1, u_2) = 0 \tag{19}$$

Very technical part

In view of (19), using the technique of $\overline{\partial}$ equations (introduced by Eskin), and a little bit luck, we can show that

$$\mu_1 = \mu_2.$$

Reconstruction of obstacle

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be an open bounded domain with smooth boundary. Assume that D is a subset of Ω such that $\overline{D} \subset \Omega$ and $\Omega \setminus \overline{D}$ is connected.

$$\begin{cases} \mathrm{div}(\mu \mathsf{Sym}(\nabla u)) - \nabla p = 0 \quad \text{in} \quad \Omega \setminus \overline{D}, \\ \mathrm{div} u = 0 \quad \text{in} \quad \Omega \setminus \overline{D}, \\ u = 0 \quad \text{on} \quad \partial D, \\ u = f \in H^{1/2}(\partial\Omega) \quad \text{on} \quad \partial\Omega, \end{cases} \tag{20}$$

where

$$\int_{\partial\Omega} f \cdot \mathbf{n} ds = 0.$$

For this problem, we are interested in the determination of D from the set of measurements

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Following is a reconstruction method proposed by Heck-Uhlmann-W.

Ideas

Energy inequalities

Let (u_0, p_0) be a solution of

$$\begin{cases} \mathrm{div}(\mu S(\nabla u_0)) - \nabla p_0 = 0 \quad \text{in} \quad \Omega, \\ \mathrm{div} u_0 = 0 \quad \text{in} \quad \Omega, \\ u_0 = f \quad \text{on} \quad \partial\Omega. \end{cases} \tag{21}$$

(solution to the system without obstacle). Then

$$\begin{aligned} &\int_D |\mathrm{Sym}(\nabla u_0)|^2 dx \\ &\leq \langle \sigma(u,p)\mathbf{n} - \sigma(u_0,p_0)\mathbf{n}, f\rangle \\ &\leq C\big(\int_D |\mathrm{Sym}(\nabla u_0)|^2 dx + \int_D |u_0|^2 dx \big). \end{aligned}$$

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What kind of special solutions are useful?

Try complex geometrical optics solutions!

CGO solutions

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As before, using the new system:

$$P\begin{pmatrix} w \\ g \end{pmatrix} := \Delta \begin{pmatrix} w \\ g \end{pmatrix} + A_1(x)\begin{pmatrix} \nabla g \\ \mathrm{div} w \end{pmatrix} + A_0(x)\begin{pmatrix} w \\ g \end{pmatrix} = 0. \quad (23)$$

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$(u_0 = \mu^{-1/2}w + \mu^{-1}\nabla g - (\nabla\mu^{-1})g)$

Look for

$$u_0 = e^{-\phi/h}v \quad (h \ll 1)$$

for appropriate phase function ϕ and amplitude v. (complex WKB)

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Let us denote

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φ is required to satisfy the following condition: if $a(x,\xi) = |\xi|^2 - |\varphi_x'|^2$ and $b(x,\xi) = 2\varphi_x' \cdot \xi$, then

$$\{a,b\}(x,\xi) = 0 \quad \text{when} \quad a(x,\xi) = b(x,\xi) = 0.$$

(limiting Carleman weight)

$u_{t,h} = e^{t/h} u_0 = e^{(t-\varphi)/h} e^{i\psi/h} v$ is also a solution.

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and measure $\sigma(u,p)\mathbf{n}|_{\partial\Omega}$ (output).

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$$E(t, h) = \langle \sigma(u, p)\mathbf{n} - \sigma(u_0, p_0)\mathbf{n}, f_{t,h} \rangle.$$

$E(t, h)$ is completely determined by boundary measurements.

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<u>Behavior of $u_{t,h}$</u>

$u_{t,h} \uparrow \infty$ as $h \to 0$ for $\varphi > t$

$u_{t,h} \downarrow 0$ as $h \to 0$ for $\varphi < t$

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So we can determine whether the level surface $\varphi = t$ touches ∂D from the behavior of $E(t,h)$.

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$\varphi = \log|x - x_0|$ and $t = \log s$: level surfaces are spheres or circles $\Rightarrow$ determine some non-convex parts of D (Ide, Isozaki, Nakata, Siltanen, Uhlmann).

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Both cases work for our problem here.

Problems in 2 dim

One is able to determine more information of D by using CGO with more general phase functions. Why? We have rich conformal structures in 2 dim.

We can construct CGO with general phases for systems like

$$PU := \Delta_x U + A_1(x)\partial_{x_1} U + A_2(x)\partial_{x_2} U + Q(x)U = 0 \quad \text{in} \quad \Omega, \tag{25}$$

where $\Delta_x = \partial_{x_1}^2 + \partial_{x_2}^2$ and A_1, A_2, Q are $n \times n$. This system includes the conductivity equation, the isotropic elasticity, the Stokes system etc.

How it works?

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Let $y = \rho(x) = y_1(x_1, x_2) + iy_2(x_1, x_2)$ be a conformal map. Define $U(x) = V(y(x))$, we have

$$\begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \end{pmatrix} U = J(x) \begin{pmatrix} \partial_{y_1} \\ \partial_{y_2} \end{pmatrix} V \Big|_{y=\rho(x)} \quad \text{and} \quad \Delta_x U = \Delta_y V |\rho'(x)|^2,$$

where

$$J(x) = \begin{pmatrix} \partial_{x_1} y_1 & \partial_{x_1} y_2 \\ \partial_{x_2} y_1 & \partial_{x_2} y_2 \end{pmatrix}.$$

How it works?

Let $y = \rho(x) = y_1(x_1, x_2) + iy_2(x_1, x_2)$ be a conformal map. Define $U(x) = V(y(x))$, we have

$$\begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \end{pmatrix} U = J(x) \begin{pmatrix} \partial_{y_1} \\ \partial_{y_2} \end{pmatrix} V \Big|_{y=\rho(x)} \quad \text{and} \quad \Delta_x U = \Delta_y V |\rho'(x)|^2,$$

where

$$J(x) = \begin{pmatrix} \partial_{x_1} y_1 & \partial_{x_1} y_2 \\ \partial_{x_2} y_1 & \partial_{x_2} y_2 \end{pmatrix}.$$

The form of P does not change under the conformal transform.

So we can first construct CGO with linear phase, i.e., $y_1 + iy_2$, and perform conformal transform. Thus we have CGO with phases $y_1(x_1, x_2) + iy_2(x_1, x_2) = \varphi(x) + i\psi(x)$.

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"*Scientists discover the world that exists; engineers create the world that never was.*" – Theodore Von Karman merci beaucoup