About the influence of the wind on the oceanic motion

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A mathematical model for large-scale oceanic motions

▶ The homogeneous incompressible Navier-Stokes equations with Coriolis force

$$\partial_t u + (u \cdot \nabla) u + \nabla p = \mathcal{F} + u \wedge \Omega \,,$$
$$\nabla \cdot u = 0 \,,$$

- $\mathcal{F}$ friction force ;
- Ω vertical component of the Earth rotation vector ;
- p pressure (defined as the Lagrange multiplier associated to the incompressibility constraint).

Using the f-plane approximation at mid-latitudes

$$D_x \Omega = 0.$$

Modelling turbulent effects by some anisotropic viscosity

$$\mathcal{F} = A_h \Delta_h u + A_z \partial_{zz} u$$

▶ Boundary conditions at the surface and at the bottom

The ocean/atmosphere coupling

- rigid lid approximation (no free surface as first approximation)
- stress tensor to account for the effect of the wind

$$\begin{aligned} u_{3|z=1} &= 0, \\ \partial_z u_{h|z=1} &= \Sigma\,. \end{aligned} \qquad (1)$$

The ocean/Earth crust coupling

- flat bottom approximation (topographic effects to be studied later)
- braking condition to account for the fluid/structure interaction

$$u_{|z=0} = 0. \qquad (2)$$

The small Rossby number asymptotics

Typical scales

- Length scales : $L \sim 100km$, $D \sim 4km$
- Velocity scale $U \sim 5cm/s$
- Viscosity $A_h \sim 10^7 cm^2/s$, $A_z \sim 10cm^2/s$
- Density of the fluid $\rho \sim 10^3 kg/m^3$

Nondimensional parameters

$$\frac{U}{L\Omega} = \varepsilon \sim 5 \times 10^{-3}, \quad \frac{A_h}{\rho UL} \sim 1, \quad \frac{LA_z}{\rho UD^2} = \nu \sim 10^{-3}$$

The singular penalization problem

$$\begin{aligned} \partial_t u + (u \cdot \nabla)u + \frac{1}{\varepsilon} e_3 \wedge u - \Delta_h u - \nu \partial_{zz} u + \nabla p = 0 \\ \nabla \cdot u = 0 \, , \end{aligned} \tag{3}$$

The Coriolis operator

$$L : u \mapsto e_3 \wedge u + \nabla p \quad \text{with} \quad \nabla \cdot (e_3 \wedge u + \nabla p) = 0$$

▶ The quasigeostrophic motion

The geostrophic constraint

$$e_3 \wedge \bar{u} = -\nabla p, \quad \nabla \cdot \bar{u} = 0$$

- motion that does not depend on $z \Rightarrow$ Taylor-Proudman columns
- constant height of water $\Rightarrow$ 2D motion without constraint

The formal evolution equation

$$\partial_t \bar{u}_h + (\bar{u}_h \cdot \nabla_h)\bar{u}_h + \nabla_h p = 0$$

- equation that does not take into account the effect of small scales
- motion that is not compatible with the boundary conditions

▶ Wave and envelope equations

On $(\mathrm{Ker}\, L)^\perp$, the dominant process is governed by the Coriolis operator

$$\partial_t u + \frac{1}{\varepsilon} L u = O(1)$$

It describes the propagation of **Poincaré waves**

- propagating with a speed of order ε^{-1} ;
- having the dispersion law $\lambda_k = -k_3\pi / \sqrt{|k_h|^2 + (\pi k_3)^2}$;
- carrying a finite energy.

Their slow evolution is given by the **envelope equations** obtained formally by some filtering method based

- on the decomposition of any field of

  $$V_0 = \{u \in L^2(\omega) \,/\, \nabla \cdot u = 0 \text{ and } u_{3|z=0} = u_{3|z=1} = 0\}$$

  on the eigenmodes of L ;
- on a precise study of resonances.

▶ Diagonalization of the Coriolis operator

Proposition. There exists an hilbertian basis (N^k) of V_0 constituted of eigenvectors of the linear penalization

$$LN^k = i\lambda_k N^k \textbf{ with } \lambda_k = -k_3\pi/\sqrt{|k_h|^2 + (\pi k_3)^2}\,.$$

For instance the family (N^k) defined by

$$N^k = \exp(ik_h \cdot x_h) \begin{pmatrix} n_1(k)\cos(\pi k_3 z) \\ n_2(k)\cos(\pi k_3 z) \\ n_3(k)\sin(\pi k_3 z) \end{pmatrix}$$

$$\text{with } \begin{cases} n_h(k) = \dfrac{1}{2\pi|k_h|}(ik_h^\perp + k_h\lambda_k), \quad n_3(k) = -i\dfrac{|k_h|\lambda_k}{2\pi^2 k_3} \text{ if } k_h \neq 0 \\ n_1(k) = \dfrac{\mathrm{sgn}(k_3)}{2\pi}, \quad n_2(k) = \dfrac{i}{2\pi}, \quad n_3(k) = 0 \text{ if } k_h = 0 \end{cases}$$

(see [Chemin, Desjardins, Gallagher, Grenier] for a detailed study)

The boundary operator

▶ Ansatz and equation for the decay rate

The vector fields N_k do not satisfy the boundary conditions (1)(2). To **match the horizontal boundary conditions** in (2), we seek a particular solution to (3) in the form

$$\Phi^{k_h,\mu}(t,x_h,z) = \varphi(k_h,\mu)\exp(ik_h\cdot x_h)\exp\left(i\frac{t}{\varepsilon}\mu\right)\exp\left(-\lambda\frac{z}{\sqrt{\varepsilon\nu}}\right)$$

The balance between forces in the boundary layers states

$$\begin{aligned} i\mu\varphi_1 + \lambda^2\varphi_1 - \varepsilon k_h^2\varphi_1 - \varphi_2 + \varepsilon\nu\frac{k_1k_2\varphi_1 - k_1^2\varphi_2}{\lambda^2 - \varepsilon\nu k_h^2} = 0,\\ i\mu\varphi_2 + \lambda^2\varphi_2 - \varepsilon k_h^2\varphi_2 + \varphi_1 + \varepsilon\nu\frac{-k_1k_2\varphi_2 + k_2^2\varphi_1}{\lambda^2 - \varepsilon\nu k_h^2} = 0,\\ \sqrt{\varepsilon\nu}(ik_1\varphi_1 + ik_2\varphi_2) - \lambda\varphi_3 = 0\,. \end{aligned}$$

▸ Interpretation in terms of dynamical system

The existence of exponentially decaying solutions is related to the fact that the matrix

$$A_\lambda(\mu, k_h) = \begin{pmatrix} \mu & i \\ -i & \mu \end{pmatrix} + o(1)$$

has eigenvalues with non zero real parts.

- if $|\mu| \neq 1$, $A_\lambda(\mu, k_h)$ perturbation of an hyperbolic system
  $\Rightarrow A_\lambda(\mu, k_h)$ hyperbolic system
- if $|\mu| = 1$, $A_\lambda(\mu, k_h)$ perturbation of a degenerate system
  - the system is hyperbolic (with a very small eigenvalue) if $k_h \neq 0$
  In that case, pressure cannot be neglected !
  - the system is non hyperbolic if $k_h = 0$ (no perturbation)
  In that case, boundary effects are no more localized !

▶ The Ekman layers and profiles

- If $k_h \neq 0$ or $|\mu| \neq 1$

$\Phi^{k_h,\mu}$ is a linear combination of

$$\Phi^{k_h,\mu,\pm} = \begin{pmatrix} w_\lambda(\mu, k_h)) \\ \frac{\sqrt{\varepsilon\nu}}{\lambda} ik_h \cdot w_\lambda(\mu, k_h)) \end{pmatrix} \exp(ik_h \cdot x_h) \exp(i\mu\frac{t}{\varepsilon}) \exp\left(-\lambda\frac{z}{\sqrt{\varepsilon\nu}}\right)$$

where

$$\det(A_\lambda + i\lambda^2 Id) = 0 \text{ and } A_\lambda w_\lambda = -i\lambda^2 w_\lambda.$$

- Usual Ekman layers if $\lambda = O(1)$
- Anomalous boundary layers if $\lambda = O((\varepsilon\nu)^{1/4})$ $(|\mu| = 1)$

- If $k_h = 0$ and $|\mu| = 1$

$\Phi^{k_h,\mu}$ is a linear combination of a usual Ekman boundary term and of some destabilization profile

$$\begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix} \exp\left(\pm i\frac{t}{\varepsilon}\right)$$

▶ The Ekman velocity

The vector fields $\Phi^{k_h,\mu,\pm}$ do not belong to V_0.
To **restore the zero-flux condition**, we introduce some corrector.

$$\delta\Phi_3^{k_h,\mu,\pm} = \frac{\sqrt{\varepsilon\nu}}{\lambda^\pm(\mu,k_h)} ik_h \cdot w^\pm(\mu,k_h)\exp(ik_h\cdot x_h)\exp(i\mu\frac{t}{\varepsilon}) \times \left(z\left(1-\exp\left(-\frac{\lambda^\pm(\mu,k_h)}{\sqrt{\varepsilon\nu}}\right)\right)-1\right)$$

$$\delta\Phi_h^{k_h,\mu,\pm} = -\frac{\sqrt{\varepsilon\nu}}{\lambda^\pm(\mu,k_h)}\frac{k_h k_h\cdot w^\pm(\mu,k_h)}{|k_h|^2}\exp(ik_h\cdot x_h)\exp(i\mu\frac{t}{\varepsilon}) \times \left(1-\exp\left(-\frac{\lambda^\pm(\mu,k_h)}{\sqrt{\varepsilon\nu}}\right)\right)$$

$\delta\Phi_3^{k_h,\mu,\pm}$ is the **Ekman transpiration velocity**.
It will be responsible for global circulation in the whole domain

- of order $(\varepsilon\nu)^{\frac{1}{2}}$
- but not limited to the boundary layer.

▶ Propagation of he horizontal boundary data

Proposition. With the previous notations, for any $\mu \neq \pm 1$, the family $(\Psi^{k_h,\mu,\pm})$ of V_0 defined by

$$\Psi^{k_h,\mu,\pm} = \Phi^{k_h,\mu,\pm} + \delta\Phi^{k_h,\mu,\pm}$$

- **is constituted of quasi-eigenvectors of the linear operator**

$$L\Psi^{k_h,\mu,\pm} + \varepsilon\Delta_h\Psi^{k_h,\mu,\pm} + \varepsilon\nu\partial_{zz}\Psi^{k_h,\mu,\pm} = -i\mu\Psi^{k_h,\mu,\pm} + O(\sqrt{\nu})$$

- **is such that $(\Psi^{k_h,\mu,\pm}_{h|z=0})_{k_h,\pm}$ is an hilbertian basis of $L^2(\omega_h, \mathbb{R}^2)$.**

(see [Dalibard, Saint-Raymond] for a more general statement including the case $|\mu| = 1$)

The high rotating limit

▶ The coupling between Poincaré waves

Obtained formally by filtering the waves $v^\varepsilon = \exp(tL/\varepsilon)u^\varepsilon$

$$\partial_t v^\varepsilon - \Delta_h v^\varepsilon - \nu \exp(tL/\varepsilon)\partial_{zz} \exp(-tL/\varepsilon)v^\varepsilon$$
$$= -\exp(tL/\varepsilon)\mathbb{P}\nabla_x \cdot (\exp(-tL/\varepsilon)v^\varepsilon \otimes \exp(-tL/\varepsilon)v^\varepsilon)$$

One has then to characterize the resonances

$$\lambda_k = \lambda_l + \lambda_m \text{ with } k = l + m$$

Some properties of the limiting coupling

- there are few resonances (no non trivial resonance generically),
- the nonlinear term corresponding to the triplet $(0, k, k)$ for $k \neq 0$ is identically zero, meaning that the **mean motion decouples from the oscillating part of the system**.

(see [Chemin, Desjardins, Gallagher, Grenier] for instance)

▶ The boundary effects

Do not play any role in the nonlinear process since they are localized in the vicinity of the surface.

Result only from the non commutation between $\frac{1}{\varepsilon}L$ and $\nu\partial_{zz}$

$$\nu \exp(tL/\varepsilon)\partial_{zz} \exp(-tL/\varepsilon)$$

(in some suitable functional space depending on the boundary conditions)

Characterization of the **Ekman pumping**

- at the bottom, damping of the interior motion (friction term)
- at the surface, no contribution (scaling of the wind-stress)

(see [Desjardins, Grenier] or [Masmoudi] for instance)

▶ Statement of the result

Theorem. Let $u_0 \in V_0$, and $\Sigma \in L^2_{loc}(\mathbb{R}^+, L^2(\omega_h))$. For all $\varepsilon > 0$, denote by u^ε a weak solution of (1)(2)(3).

Then (u^ε) converges weakly in $L^2_{loc}(\mathbb{R}^+ \times \omega)$ to a limit u such that

- **if $\lim_{\varepsilon\to 0} \frac{\nu}{\varepsilon} = +\infty$, $u = 0$**
- **if $\lim_{\varepsilon\to 0} \frac{\nu}{\varepsilon} = \beta < +\infty$, $u = (u_h, 0)$ is the solution of the two dimensional Navier-Stokes equations with Ekman pumping**

$$\begin{cases} \partial_t u_h + (u_h \cdot \nabla_h) u_h - \Delta_h u_h + \dfrac{\beta}{\sqrt{2}} u_h + \nabla_h p = 0, \\ \nabla_h \cdot u_h = 0\,, \\ u_{h|t=0} = \displaystyle\int_0^1 u_{0,h} dz\,. \end{cases} \tag{4}$$

▶ Weak vs strong convergence methods

We present here a method which is based

- on some uniform energy estimate,
- on 2-scales techniques (to deal with boundary layers),
- on compensated compactness arguments (to handle the nonlinearity).

Simple strategy $\Rightarrow$ easily extended to more complex problems
(variable density, variable Coriolis force)

Weak convergence result $\Rightarrow$ focuses on the mean motion
(no description of the envelope equations)

In both cases, boundary layers do not appear in the approximation, since they have negligible L^2 norm and the convergence holds in L^2.

The weak convergence method

▶ The weak formulation

Start from the weak form of (1)(2)(3) : for any test function $\Phi \in \mathrm{Ker}\, L$,

$$\int u_0 \cdot \Phi_h dx + \int_0^t \int \left(u_h^\varepsilon \cdot \partial_t \Phi_h + u_h^\varepsilon \otimes u_h^\varepsilon : \nabla_h \Phi_h - \nabla_h u_h^\varepsilon : \nabla_h \Phi_h \right)(s,x) ds dx$$
$$= \nu \int_0^t \int (\partial_z u_h^\varepsilon)_{|z=0} \Phi_h(s,x_h) dx_h ds - \nu \int_0^t \int (\partial_z u_h^\varepsilon)_{|z=1} \Phi_h(s,x_h) dx_h ds \quad (5)$$

Take limits in the surface term

- if $\nu \to 0$, $\nu \int \Sigma \Phi_h dx_h \to 0$
  (scaling assumption on Σ to ensure that the energy is bounded)
- if $\nu \to \bar{\nu} > 0$, $\nu \int \Sigma \Phi_h dx_h \to \bar{\nu} \int \Sigma \Phi_h dx_h$

Boundary layer at the top does not appear (even though it exists !).

▶ The 2-scale analysis

To **relate the trace** $(\partial_z u_h^\varepsilon)_{|z=0}$ **to** u_h^ε, we use the boundary test functions $\Psi^{k_h,\mu,\pm}$ in the weak formulation of (1)(2)(3).

This process is exactly the analog of the **2-scale analysis** of N'Guetseng and Allaire.

We actually prove that the only terms that remain of order $O(\sqrt{\nu/\varepsilon})$ are

$$\nu \int (\partial_z u_h^\varepsilon)_{|z=0} \Phi^{k_h,\pm}_{h|z=0}(s, x_h) dx_h$$

and the Ekman pumping

$$\int \left(u^\varepsilon \cdot \frac{1}{\varepsilon} L \delta \Phi^{k_h,\pm} \right)(s,x) dx$$

$$= \sqrt{\frac{\nu}{2\varepsilon}} \int_0^t \int u_\varepsilon \cdot \Phi^{k_h,\pm}_{|z=0}(s,x) ds dx$$

Key arguments of the proof

- A priori estimates for $\Psi^{k_h,\pm}$

$$\Phi_h^{k_h,\mu,\pm} = O(1)_{L^\infty([0,1],H^s(\omega_h))}, \quad \Phi_h^{k_h,\mu,\pm} = O(\sqrt{\varepsilon\nu})_{L^1([0,1],H^s(\omega_h))},$$
$$\Phi_h^{k_h,\mu,\pm} = O((\varepsilon\nu)^{\frac{1}{4}-\frac{\sigma}{2}})_{H^\sigma([0,1],H^s(\omega_h))}$$
$$\delta\Phi^{k_h,\mu,\pm} = O(\sqrt{\varepsilon\nu})_{H^s(\omega)}$$

- Refined estimates on u^ε (using the Dirichlet condition at $z=0$)

$$\|u_{h|z=z_0}\|_{L^2(\omega_h)} \leq z_0^{1/2}\|\partial_z u_h\|_{L^2(\omega)} \leq C\nu^{-1/2}z_0^{1/2}\nu^{1/2}\|\partial_z u_h\|_{L^2(\omega)}$$
$$\|u_{3|z=z_0}\|_{L^2(\omega_h)} \leq z_0^{1/2}\|\partial_z u_3\|_{L^2(\omega)} \leq Cz_0^{1/2}\|\nabla_h u_h\|_{L^2(\omega)}.$$

▶ The compensated compactness

If $\lim_{\varepsilon \to 0} \frac{\nu}{\varepsilon} = \beta < +\infty$ then the Ekman pumping is an order $O(1)$ dissipation process.

The point is then to **take limits in the nonlinear terms**, i.e.

- to establish strong compactness on $\bar{u}^\varepsilon = \int_0^1 (u_h^\varepsilon, 0) dz$;
- to prove that the equation on $\bar{u}_\varepsilon$ decouples from oscillations (compensated-compactness argument).

Strong compactness of $(\bar{u}^\varepsilon)$ coming from

- the uniform spatial regularity

$$\bar{u}_\varepsilon \text{ bounded in } L^2(\mathbb{R}^+, \dot{H}^1(\omega_h))$$

- some control on the time derivative

$$\partial_t(\bar{u}^\varepsilon - O((\varepsilon\nu)^{1/4})_{L^2(\omega)}) \text{ bounded in } L^2([0,T] \times [0,1], W^{-1,1}(\omega_h))$$

- some interpolation result (see [Aubin] for instance)

Structure of the waves $u^\varepsilon - \bar{u}^\varepsilon = \partial_z W^\varepsilon$

$$\varepsilon\partial_t(\partial_z W^\varepsilon - \nabla W_3^\varepsilon) + e_3 \wedge \partial_z W_\varepsilon = r_\varepsilon,$$
$$\varepsilon\partial_t(\nabla_h \cdot W_h^{\varepsilon,\perp}) + \nabla_h \cdot W_h^\varepsilon = s_\varepsilon,$$

where r_ε and s_ε are small remainders.

Algebraic computation (see [Gallagher, Saint-Raymond] for instance)

$$\begin{aligned}(w^\varepsilon \wedge (\nabla \wedge w^\varepsilon))_h \\ &= (\varepsilon\partial_t\rho^\varepsilon \wedge \partial_z\rho^\varepsilon)_h \\ &= -\varepsilon\partial_t(\rho_3^\varepsilon\partial_z\rho_h^{\varepsilon,\perp}) + \partial_z(\rho_3^\varepsilon\varepsilon\partial_t\rho_h^{\varepsilon,\perp}) + r_\varepsilon\partial_z\rho_3^\varepsilon + s_\varepsilon\partial_z\rho_h^{\varepsilon,\perp}\end{aligned}$$

where $\rho^\varepsilon = \nabla \wedge W^\varepsilon$.

That formal computation can be made rigorous by introducing some regularization in x_h, provided that $\nu >> \varepsilon^{4/3}$.