Elliptic and parabolic regularity for discrete elliptic operators on graphs

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I. Introduction: the Euclidean case

Consider second order elliptic operators in divergence form:

$$L = -\text{div}(A\nabla)$$

where $A : \mathbb{R}^n \to M_n(\mathbb{C})$ is measurable, bounded and uniformly elliptic, *i.e.* , for some $\delta > 0$,

$$\begin{array}{lcll} |\langle A(x)\xi, \eta\rangle| & \leq & \delta^{-1} |\xi|\, |\eta| & \text{a.e. } x \in \mathbb{R}^n, \text{ for all } \xi, \eta \in \mathbb{C}^n, \\ \text{Re } \langle A(x)\xi, \xi\rangle & \geq & \delta\, |\xi|^2 & \text{a.e. } x \in \mathbb{R}^n, \text{ for all } \xi \in \mathbb{C}^n. \end{array}$$

What can be said about the elliptic and parabolic regularity theory for L ?

By elliptic regularity , we mean regularity of the solutions of

$$Lu = f$$

assuming some regularity on f.

By parabolic regularity, we mean regularity of the solutions of

$$\frac{\partial u}{\partial t} + Lu = f.$$

This parabolic regularity is related to the properties of the semigroup e^{-tL}.

If A is real-valued :

- elliptic regularity is due to De Giorgi (1957) and Morrey (1966),
- parabolic regularity is due to Nash (1958),
- Harnack inequalities are due to Moser (1961),
- the link with Gaussian estimates for the kernel of e^{-tL} is due to Aronson (1967), Fabes and Stroock (1986).

L generates a holomorphic contraction semigroup on $L^2(\mathbb{R}^n)$. If K_t is the Schwartz kernel of e^{-tL}, say that L has the Gaussian property iff there exist $C, \mu > 0$ such that

$$\begin{aligned}
|K_t(x,y)| &\leq \frac{C}{t^{\frac{n}{2}}} \exp\left(-\beta \frac{|x-y|^2}{t}\right), \\
|K_t(x,y) - K_t(x,y+h)| &\leq \frac{C}{t^{\frac{n}{2}}} \left(\frac{|h|}{t^{\frac{1}{2}} + |x-y|}\right)^{\mu} \exp\left(-\beta \frac{|x-y|^2}{t}\right), \\
|K_t(x+h,y) - K_t(x,y)| &\leq \frac{C}{t^{\frac{n}{2}}} \left(\frac{|h|}{t^{\frac{1}{2}} + |x-y|}\right)^{\mu} \exp\left(-\beta \frac{|x-y|^2}{t}\right)
\end{aligned} \tag{G}$$

for all $t > 0$, all $x, y, h \in \mathbb{R}^n$ such that $2|h| \leq t^{\frac{1}{2}} + |x-y|$.

When $A = \mathrm{Id}$, *i.e.* $L = -\Delta$,

$$K_t(x,y) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

for all $t > 0$, for all $x, y \in \mathbb{R}^n$.

(G) always holds when A is real-valued .
When A is complex-valued , (G) is true when $n=1$ or $n=2$ (Auscher, McIntosh, Tchamitchian, 1998).

When $n \geq 5$, e^{-tL} may not be $L^\infty(\mathbb{R}^n)$-bounded (Auscher, Coulhon, Tchamitchian, 1996) and therefore (G) does not hold.

(G) holds when A is complex-valued and the coefficients are Hölder continuous (Auscher, McIntosh, Tchamitchian, 1998), or when A is an L^∞ perturbation of a real-valued matrix.

A reformulation of (G): (G) is equivalent to

$$\begin{aligned} \left\| e^{-tL} \right\|_{L^2 \to L^\infty} &\leq Ct^{-n/4} \\ \left\| e^{-tL} \right\|_{L^2 \to \dot{C}^\eta} &\leq Ct^{-n/4-\eta/2} \end{aligned} \tag{1}$$

and the analogous estimates for L^*.

Indeed, the first line in (1) yield the upper bound for K_t without the Gaussian term, and this term follows from a semigroup perturbation technique due to Davies.

Then, the Hölder regularity follows from the second line in (1).

The link between elliptic and parabolic regularity for L: if $\Omega \subset \mathbb{R}^n$ is open, a weak solution of $Lu = 0$ in Ω is a function $u \in H^1(\Omega)$ such that

$$\int_\Omega A(x)\nabla u(x) \cdot \nabla \varphi(x) dx = 0$$

for all $\varphi \in \mathcal{D}(\Omega)$.

A gradient estimate due to Morrey: if A is complex-valued, there exists $C = C(n,\delta) > 0$ and $\alpha = \alpha(n,\delta)$ such that, for all weak solution u of $Lu = 0$ in Ω and all ball $B \subset \Omega$ with radius r such that $\overline{2B} \subset \Omega$,

$$\int_B \left|\nabla u(x)\right|^2 dx \leq Cr^\alpha \left\|u\right\|^2_{H^1(\Omega)}.$$

If $\alpha > n-2$, a consequence is that u is Hölder continuous. This is true when A is real-valued.

The quantities $\int_B |\nabla u(x)|^2 \, dx$ are called the Dirichlet integrals. Their growth is relevant in the study of elliptic and parabolic regularity.

Say that L satisfies the Dirichlet property iff there exist $C > 0$ and $\mu \in (0, 1)$ such that, for all ball $B(x_0, r) \subset \Omega$, all weak solution of $Lu = 0$ in $B(x_0, r)$ and all $\rho \in (0, r)$,

$$\int_{B(x_0,\rho)} |\nabla u(x)|^2 \, dx \leq C \left(\frac{\rho}{r}\right)^{n-2+2\mu} \int_{B(x_0,r)} |\nabla u(x)|^2 \, dx. \qquad (D)$$

When A is real-valued, this property always holds.

Theorem

(Auscher, Tchamitchian, 1998) If $n \geq 2$, then

$$L \text{ satisfies } (G) \Leftrightarrow L \text{ and } L^* \text{ satisfy } (D).$$

II. Approximation by a discrete operator

Let $\Omega \subset \mathbb{R}^2$ be a polygonal open set and assume that

$$\overline{\Omega} = \bigcup T_i,$$

is a triangulation of Ω, which means that:

- each T_i is a triangle,
- if $i \neq j$, T_i and T_j have disjoint interiors,
- for all i, any edge of T_i is also an edge of T_j for some $j \neq i$.

Denote by $h > 0$ the supremum of all the diameters of the T_i's.

Assume that the eccentricities of the T_i's are uniformly bounded.

Associate a graph Γ_h to this triangulation. The vertices of Γ_h are the vertices of the triangles, the edges of Γ_h are the edges of the triangles.

Consider a uniformly elliptic operator $L = -\mathrm{div}(A\nabla)$ on Ω under Dirichlet boundary condition. L is the maximal accretive operator associated with the sesquilinear form

$$\mathcal{B}(u,v) = \int_\Omega A(x)\nabla u(x) \cdot \overline{\nabla v(x)} dx, \ \ u,v \in H_0^1(\Omega).$$

We want to discretize L by the Galerkin method.

For each $x \in \Gamma_h$, let $\varphi_x : \Omega \to \mathbb{R}$ be the function such that

- $\varphi_x(y) = \delta_{xy}$,
- the restriction of φ_x to each triangle T_i is affine.

Note that

$$\sum_{x \in \Gamma_h} \varphi_x = 1.$$

Define a space of test functions by

$$V_h = \left\{ f \in H_0^1(\Omega);\ f = \sum_{x \in \Gamma_h} \lambda_x \varphi_x;\ \sum_{x \in \Gamma_h} |\lambda_x|^2 < +\infty \right\}$$

and look at the restriction of $\mathcal{B}$ to $V_h \times V_h$.

Let

$$\widetilde{u} = \sum_{x\in\Gamma} u_x\varphi_x \text{ and } \widetilde{v} = \sum_{x\in\Gamma} v_x\varphi_x \in V_h.$$

One has

$$\begin{aligned}\mathcal{B}(\widetilde{u},\widetilde{v}) &= \sum_{x\in\Gamma_h}\sum_{y\in\Gamma_h} u_x\overline{v_y}\int_\Omega A(z)\nabla\varphi_x(z)\cdot\overline{\nabla\varphi_y(z)}dz \\ &:= \sum_{x\in\Gamma_h}\sum_{y\in\Gamma_h} c_{xy}u_x\overline{v_y}\end{aligned}$$

where

$$c_{xy} = \int_\Omega A(z)\nabla\varphi_x(z)\cdot\overline{\nabla\varphi_y(z)}dz.$$

- $c_{xy} = 0$ if x and y are not neighbours,
- for all $x \in \Gamma_h$,
$$\sum_{y \sim x} c_{xy} = 0.$$
- for all $y \in \Gamma_h$,
$$\sum_{x \sim y} c_{xy} = 0.$$

As a consequence,

$$\mathcal{B}(\widetilde{u}, \widetilde{v}) = \frac{1}{2} \sum_{x \sim y} c_{xy} (u_x - u_y) \overline{(v_x - v_y)}.$$

Now, if $u : \Gamma_h \to \mathbb{C}$ and $v : \Gamma_h \to \mathbb{C}$ are such that

$$\widetilde{u} := \sum_{x \in \Gamma_h} u_x \varphi_x \in V_h \text{ and } \widetilde{v} := \sum_{x \in \Gamma_h} v_x \varphi_x \in V_h,$$

it is therefore natural to define

$$\mathcal{B}_h(u, v) := \frac{1}{2} \sum_{x \sim y} c_{xy} (u_x - u_y) \overline{(v_x - v_y)}.$$

One has

$$\|\nabla \widetilde{u}\|^2_{L^2(\Omega)} \sim \sum_{x \sim y} |u_x - u_y|^2 .$$

Ellipticity properties of $\mathcal{B}_h$:

$$\begin{aligned} |\mathcal{B}_h(u,v)| &= \left| \int_\Omega A(z)\nabla\widetilde{u}(z)\cdot\overline{\nabla\widetilde{v}(z)}dz \right| \\ &\leq \delta^{-1}\left\|\nabla\widetilde{u}\right\|_{L^2(\Omega)}\left\|\nabla\widetilde{v}\right\|_{L^2(\Omega)} \\ &\leq C\left(\sum_{x\sim y}\left|u_x-u_y\right|^2\right)^{\frac{1}{2}}\left(\sum_{x\sim y}\left|v_x-v_y\right|^2\right)^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} \mathcal{B}_h(u,u) &= \int_\Omega A(z)\nabla\widetilde{u}(z)\cdot\overline{\nabla\widetilde{u}(z)}dz \\ &\geq \delta\left\|\nabla\widetilde{u}\right\|^2_{L^2(\Omega)} \\ &\geq c\sum_{x\sim y}\left|u_x-u_y\right|^2. \end{aligned}$$

If L_h is the maximal accretive operator associated to $\mathcal{B}_h$, L_h has to be an approximation of L when $h \to 0$.

Questions:

- What can be said about the elliptic and/or the parabolic regularity properties of L_h ?
- Can one give estimates independent of h for the semigroup generated by L_h ?

In the sequel, we investigate these issues in an "abstract" context.

III. Elliptic and parabolic regularity on graphs

Let Γ be a graph (set of vertices) endowed with nonnegative weights h_{xy}, $x, y \in \Gamma$.

Say that $x \sim y$ (x and y are neighbours) iff $h_{xy} \neq 0$.

Assume that there exist $C > 0$ such that, for all $x \sim y$ and $x \sim z$,

$$C^{-1} \leq \frac{h_{xy}}{h_{xz}} \leq C.$$

Define also

$$h_x := \max_{y \sim x} h_{xy}.$$

If $x, y \in \Gamma$, a path joining x to y is a finite sequence

$$x_0 = x \sim x_1 \ldots \sim x_N = y$$

with $x_i \sim x_{i+1}$ for each i. The length of the path is $\sum_{i=0}^{N-1} h_{x_i x_{i+1}}$.

Assume that Γ is connected , *i.e.* for all $x, y \in \Gamma$, there exists a path joining x and y. The distance $d(x, y)$ is the infimum of the lengths of all paths joining x and y.

Assume also that there exists $N \geq 1$ such that, for all $x \in \Gamma$, x has at most N neighbours.

Let $f:\Gamma\rightarrow\mathbb{R}$ be a function. For all $x\in\Gamma$, the gradient of f at x is defined by

$$\nabla f(x):=\frac{1}{h_x}\left(\sum_{y\sim x}\left|f(y)-f(x)\right|^2\right)^{\frac{1}{2}}.$$

Consider a nonnegative symmetric measure μ_{xy} for all $x,y\in\Gamma$, such that $\mu_{xy}>0\Leftrightarrow x\sim y$. Set

$$m(x):=\sum_{y\sim x}\mu_{xy}.$$

For $1\leq p\leq+\infty$,

$$\left\|f\right\|_{L^p(\Gamma)}:=\left(\sum_{x\in\Gamma}\left|f(x)\right|^p m(x)\right)^{\frac{1}{p}}$$

and

$$\left\|f\right\|_{W^{1,p}(\Gamma)}:=\left\|f\right\|_{L^p(\Gamma)}+\left\|\nabla f\right\|_{L^p(\Gamma)}.$$

Definition of the operator L: given coefficients $c_{xy} \in \mathbb{C}$ for all $x \sim y$, such that there exists $C > 0$ with

$$|c_{xy}| \leq C \text{ for all } x \sim y,$$

define

$$\mathcal{B}(u,v) := \sum_{x \in \Gamma} \sum_{y \sim x} c_{xy} \frac{u(y)-u(x)}{h_{xy}} \overline{\frac{v(y)-v(x)}{h_{xy}}} \mu_{xy}$$

for $u, v \in W^{1,2}(\Gamma)$. One has

$$|\mathcal{B}(u,v)| \leq C \left\| \nabla u \right\|_{L^2(\Gamma)} \left\| \nabla v \right\|_{L^2(\Gamma)}.$$

Assume that , for some $c > 0$,

$$\mathcal{B}(u,u) \geq c \left\| \nabla u \right\|_{L^2(\Gamma)}^2 \text{ for all } u \in W^{1,2}(\Gamma).$$

This does not imply that each c_{xy} is bounded from below by c.

Define L as the maximal accretive operator on $L^2(\Gamma)$ such that

$$\langle Lu, v \rangle_{L^2(\Gamma)} = \mathcal{B}(u,v)$$

for all $u \in \mathcal{D}(L)$ and all $v \in L^2(\Gamma)$. One has

$$\begin{array}{rcl} \left| \langle Lu, v \rangle \right| & \leq & C \left\| \nabla u \right\|_{L^2(\Gamma)} \left\| \nabla v \right\|_{L^2(\Gamma)}, \\ \langle Lu, u \rangle & \geq & c \left\| \nabla u \right\|_{L^2(\Gamma)}^2. \end{array}$$

For all $x \in \Gamma$ and all $r > 0$, let $B(x,r) := \{y \in \Gamma;\ d(y,x) < r\}$ and $V(x,r) := m(B(x,r))$.

If B is a ball, set

$$\partial B = \{x \in B;\ \exists y \sim x \text{ with } y \notin B\}.$$

and

$$\overset{\circ}{B} = B \setminus \partial B.$$

Let $B = B(x_0, R)$ be a ball and $u \in W^{1,2}(B)$. Say that u is harmonic in B iff

$$\sum_{x \in \overset{\circ}{B}} \sum_{y \in \overset{\circ}{B}} c_{xy} \frac{u(y) - u(x)}{h_{xy}} \overline{\frac{v(y) - v(x)}{h_{xy}}} \mu_{xy} = 0$$

for all $v \in W^{1,2}(B)$ supported in $\overset{\circ}{B}$.

The elliptic regularity property for L is formulated in the following way:

Say that L satisfies the De Giorgi property iff there exist $C, \mu > 0$ such that, for any ball $B = B(x_0, R) \subset \Gamma$, any $\rho \in (0, R)$ and any harmonic function u in B,

$$\|\nabla u\|^2_{L^2(B(x_0,\rho))} \leq C \frac{V(x_0,\rho)}{V(x_0,R)} \left(\frac{\rho}{R}\right)^{2(\mu-1)} \|\nabla u\|^2_{L^2(B(x_0,R))}. \qquad (DG)$$

An assumption about the volume of the balls: there exist $C, D > 0$ such that, for all $x \in \Gamma$ and all $r > 0$,

$$C^{-1} r^D \leq V(x, r) \leq C r^D. \qquad (V_D)$$

This means that (Γ, d, m) is an Ahlfors-regular space.

We also assume an L^2-Poincaré inequality on balls: there exists $C > 0$ such that, for all ball B with radius r and all $f \in W^{1,2}(B)$,

$$\|f - f_B\|^2_{L^2(B)} \leq C r^2 \|\nabla f\|^2_{L^2(B)} \qquad (P)$$

where

$$f_B := \frac{1}{V(B)} \sum_{x \in B} f(x) m(x).$$

A formulation of the parabolic regularity for L: say that L satisfies the parabolic regularity property iff, for some $C, \eta > 0$,

$$\begin{aligned} \left\| e^{-tL} \right\|_{L^2 \to L^\infty} &\leq Ct^{-D/4} \\ \left\| e^{-tL} \right\|_{L^2 \to \dot{C}^\eta} &\leq Ct^{-D/4-\eta/2} \end{aligned} \qquad (PR)$$

and the analogous estimates for L^*.

One has:

Theorem

(R, 2008) Assume that Γ satisfies (V_D) and (P). Let L be a second order uniformly elliptic operator as before. Then, we have the following equivalence:

1. *L and L^* satisfy the De Giorgi property,*
2. *L satisfies the parabolic regularity property.*

The "Gaussian" estimates for K_t: if L satisfies the parabolic regularity property, one has

$$|K_t(x,y)| \leq \frac{C}{t^{D/2}} \exp\left(-\beta \frac{d(x,y)}{h^*_{xy}}\right)$$

if $t < C' h^*_{xy} d(x,y)$, and

$$|K_t(x,y)| \leq \frac{C}{t^{D/2}} \exp\left(-\beta \frac{d^2(x,y)}{t}\right)$$

if $t \geq C' h^*_{xy} d(x,y)$. Moreover

$$|K_t(x,y) - K_t(x,z)| \leq \frac{C}{t^{D/2}} \left(\frac{d(x,y)}{\sqrt{t}}\right)^{\mu}.$$

A particular case: if

- $h_{xy} = 1$ for all $x \sim y$,
- $c_{xy} = c_{yx}$,
- $c_{xy} \geq \delta > 0$ for all $x \sim y$,

Under these assumptions, it was proved by Delmotte (1999) that L satisfies the parabolic regularity property.

In general, the assumption

$$c_{xy} \geq \delta > 0 \text{ for all } x \sim y$$

is not satisfied (Galerkin method).

If $D=2$, then L and L^* satisfy the De Giorgi property (T. Rey, PhD, 2004). As a consequence, L has the parabolic regularity property.

IV. Estimates for the square root of L

From now on, we always assume that

- $h_{xy} = 1$ for all $x \sim y$,
- $c_{xy} = c_{yx}$,
- $c_{xy} \geq \delta > 0$ for all $x \sim y$.

We also assume the doubling property for the volume of balls:

$$V(x, 2r) \leq CV(x, r). \qquad (D)$$

One has

$$\left\| L^{1/2} f \right\|_2 \sim \|\nabla f\|_2 .$$

We want to give an L^p version of this comparison.

A general fact: if

$$\|\nabla f\|_p \leq C_p \left\| L^{1/2} f \right\|_p ,$$

for all f and for some $p \in (1, +\infty)$, then

$$\left\| L^{1/2} f \right\|_q \leq C_q \|\nabla f\|_q .$$

where $\frac{1}{p} + \frac{1}{q} = 1$. The converse is unclear (and probably false).

Theorem

(R, 2000) Assume that (D) and (P) hold. Then, for all $p \in (1,2]$,

$$\|\nabla f\|_p \leq C_p \left\| L^{1/2} f \right\|_p ,$$

and, for all $q \in [2, +\infty[$,

$$\left\| L^{1/2} f \right\|_q \leq C_q \|\nabla f\|_q .$$

The Riesz transform $\nabla L^{-1/2}$ *is* L^p*-bounded for* $1 < p \leq 2$.

We characterize the L^p-boundedness of $\nabla L^{-1/2}$ for $p > 2$ in terms of reverse Hölder inequalities for the gradient of harmonic functions.

Theorem

(Badr, R, 2007) Assume that (D) and (P) hold. Then, there exists $p_0 \in (2, +\infty]$ such that, for all $q \in (2, p_0)$, the following two conditions are equivalent:

1. *$\nabla L^{-1/2}$ is L^p-bounded for all $p \in (2, q)$,*
2. *for all $p \in (2, q)$, there exists $C_p > 0$ such that, for all ball $B \subset \Gamma$, all function u harmonic in $32B$,*

$$\left(\frac{1}{V(B)}\sum_{x\in B}\left|\nabla u(x)\right|^p m(x)\right)^{\frac{1}{p}} \leq C_p\left(\frac{1}{V(16B)}\sum_{x\in 16B}\left|\nabla u(x)\right|^2 m(x)\right)^{\frac{1}{2}}. \qquad (RH_p)$$

This extends a result of Shen when $L = -\mathrm{div}(A\nabla)$ in $\mathbb{R}^n$.

It is easy to see (Gehring's type argument) that there exists $\varepsilon > 0$ such that (RH_p) holds for all $p \in (2, 2+\varepsilon)$. As a consequence:

Theorem

(Badr, R, 2007) Assume that (D) and (P) hold. Then there exists $\varepsilon > 0$ such that, for all $p \in (2-\varepsilon, 2+\varepsilon)$,

$$\left\| L^{1/2} f \right\|_p \sim \left\| \nabla f \right\|_p .$$

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