Measure concentration, functional inequalities, and curvature of metric measure spaces

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Luminy, March 2008

circle of ideas

between analysis, geometry and probability theory

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- **concentration of measure phenomenon**

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theory of optimal transportation

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$$K \cap F$$

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using a concentration property
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isoperimetric inequality on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$

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normalized volume $\mu(\cdot) = \dfrac{\mathrm{vol}\,(\cdot)}{\mathrm{vol}\,(\mathbb{S}^n)}$

if $\mu(A) \geq \frac{1}{2}$

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$$\mu(A_r) \approx 1$$

application by V. Milman

m (normalized) Haar measure on $\mathcal{O}_{n+1}$

$$m\left(\{T\in\mathcal{O}_{n+1};\, Tx\in A\}\right) = \mu(A), \quad x\in\mathbb{R}^{n+1},\, A\subset\mathbb{S}^n$$

$\|\cdot\|$ gauge of K

show that for $\dim(E)\geq\delta(\varepsilon)\log n$

$$m\left(\left\{T\in\mathcal{O}_{n+1};\ \frac{|z|}{1+\varepsilon}\leq\|Tz\|\leq\frac{|z|}{1-\varepsilon},\ z\in E\right\}\right) > 0$$

$$F = \mathrm{Im}\, T_{|E}\,, \quad \mathcal{E} = T(B_E)$$

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variety of examples and tools

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model : Gaussian measure

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Gauss space : curvature 1 dimension ∞

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infinite dimensional analysis

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notion of curvature in metric measure spaces

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(PDE and calculus of variations viewpoint)

geometric description : Brunn-Minkowski inequality

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Prékopa-Leindler theorem

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$u = \chi_A, v = \chi_B$, multiplicative form of Brunn-Minkowski

$$\mathrm{vol}_n\big(\theta A + (1-\theta) B\big) \ \geq \ \mathrm{vol}_n(A)^\theta \mathrm{vol}_n(B)^{1-\theta}$$

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$$\gamma(A_r) \;\geq\; 1 - 2\,e^{-r^2/4}, \quad \gamma(A) \;\geq\; \frac{1}{2}$$

extension : $\quad d\mu = e^{-V}dx, \quad V'' \geq c > 0$

measure/information theoretic description :

transportation cost inequality

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relative entropy

M. Talagrand 1996

concentration via the transportation cost inequality (K. Marton)

$$A, B \subset \mathbb{R}^n, \quad d(A,B) \geq r > 0$$

$$\gamma_A = \gamma(\cdot\,|A), \quad \gamma_B = \gamma(\cdot\,|B)$$

$$W_2(\gamma_A, \gamma_B) \leq \left(\log \frac{1}{\gamma(A)}\right)^{1/2} + \left(\log \frac{1}{\gamma(B)}\right)^{1/2}$$

$$\frac{r}{\sqrt{2}} \leq W_2(\gamma_A, \gamma_B) \leq \left(\log \frac{1}{\gamma(A)}\right)^{1/2} + \left(\log \frac{1}{\gamma(B)}\right)^{1/2}$$

$\gamma(A) \geq 1/2, \quad B =$ complement of A_r

$$\gamma(A_r) \geq 1 - e^{-r^2/4}, \quad r \geq r_0$$

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A. Stam 1959, L. Gross 1975

concentration via the logarithmic Sobolev inequality (I. Herbst)

$F : \mathbb{R}^n \to \mathbb{R}$ 1-Lipschitz, $\int F \, d\gamma = 0$

$f = e^{\lambda F} / \int e^{\lambda F} d\gamma, \quad \lambda \in \mathbb{R}$

differential inequality on $\int e^{\lambda F} d\gamma$

$$\int e^{\lambda F} d\gamma \leq e^{\lambda^2/2}, \quad \lambda \in \mathbb{R}$$

$$\gamma(\{F < r\}) \geq 1 - e^{-r^2/2}, \quad r > 0$$

Brunn-Minkowski inequality

logarithmic Sobolev inequality

transportation cost inequality

hierarchy

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tools to establish these inequalities

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parametrisation methods

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equivalent to a curvature condition

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◇ manifolds with weights

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parametrisation by optimal transportation

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Riemannian geometry of $(\mathcal{P}_2, W_2)$ F. Otto (2001), C. Villani (2005)

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(manifold case R. McCann 1995)

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non-smooth analysis, PDE methods

mass transportation method

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optimal parametrisation and entropy

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H relative entropy, W_2 Wasserstein distance

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in a metric measure space (length space) (X, d, μ)

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J. Lott - C. Villani, K.-Th. Sturm 2006