On a nonlocal problem arising from phytoplankton blooms: Incomplte mixing and competition for light

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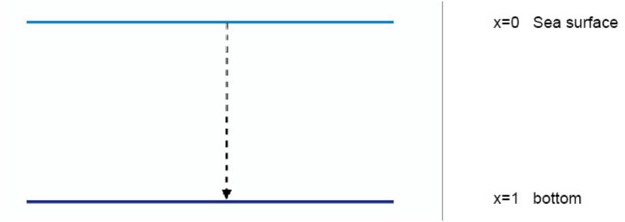
Consider the following mathematical model
[Huisman, American Naturalist 1999]

$$
\begin{aligned}
&(u_i)_t = D_i(u_i)_{xx} + (g_i(I(x,t)) - d_i)u_i,\\
&(u_i)_x(0,t) = 0,\ (u_i)_x(1,t) = 0,\\
&u_i(x,0) = u_i(x) \geq 0, i = 1, 2, \cdots, n,\\
&g_i(I) = \frac{m_i I}{a_i + I},\\
&I(x,t) = I_0 e^{-k_0 x} \exp\left(- \int_0^x (k_1 u_1(s,t) + \cdots + k_n u_n(s,t))\ ds \right).
\end{aligned}
$$

Assume the nutrient are abundance, species compete for light.

$I(x,t) =$ light intensity.
$u_i(x,t) =$ biomass of the i-th phytoplankton species.



Without loss of generality, we assume $I_0 = 1$, $D_1 = 1$. Consider the steady state problem of single population.

$$\begin{cases} -u''(x) = \left[g\left(e^{-k_0 x} e^{-k \int_0^x u(s)\, ds} \right) - d \right] u(x) \\ u'(0) = 0, \; u'(1) = 0 \end{cases} \tag{1}$$

Theorem 1

Problem (1) has a unique positive solution for $d \in (0, d_*)$, where

$$-d_* = \lambda_1 \left(-g \left(e^{-k_0 x} \right) \right),$$

which is the first eigenvalue of

$$\begin{cases} -\phi'' + \left[-g \left(e^{-k_0 x} \right) \right] \phi = \lambda \phi \\ \phi'(0) = 0, \ \phi'(1) = 0. \end{cases}$$

It has no positive solution if $d \notin (0, d_*)$.

Moreover, if we denote the unique positive solution by u_d, then

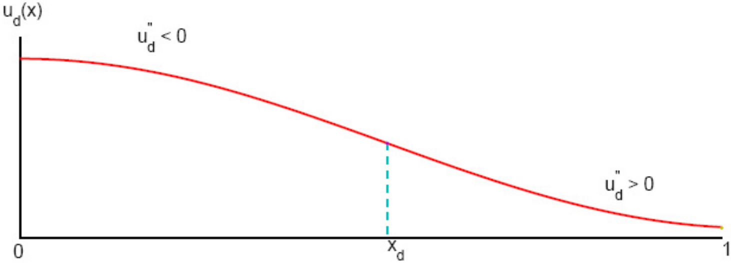
(i) $d \mapsto u_d$ is continuous from $(0, d_*)$ to $C^2([0,1])$.

(ii) $0 < d_1 < d_2 < d_* \Rightarrow u_{d_1}(0) > u_{d_2}(0)$.

(iii) $u_d \to \infty$ uniformly on $[0,1]$ as $d \to 0$.

(iv) For each d, $0 < d < d_*$, $\exists\, x_d \in (0,1)$ s.t.

$$u_d''(x) < 0,\ 0 < x < x_d$$
$$u''(x) > 0,\ x_d < x < 1$$



Proof

From standard bifurcation theory, (1) has an unbounded branch of positive solutions

$$\Gamma = \{(d, u)\} \subseteq \mathbb{R}^1 \times C^1([0,1])$$

bifurcates from trivial solution $\{(d, 0)\}$ at $(d_*, 0)$.
If (d, u) is a positive solution, then

$$\begin{aligned}
&-u'' + \left[-g\left(e^{-k_0 x} e^{-k\int_0^x u(s)\,ds}\right)\right] u = -du, \\
&-d = \lambda_1\left[-g\left(e^{-k_0 x} e^{-k\int_0^x u(s)\,ds}\right)\right] < \lambda_1(0) = 0 \\
&-d > \lambda_1\left(-g\left(e^{-k_0 x}\right)\right) = -d_*
\end{aligned}$$

$\Rightarrow -d \in (-d_*, 0) \Rightarrow 0 < d < d_*$

Therefore (1) has no positive solution when $d \notin (0, d_*)$.

Next we show Γ becomes unbounded

$$\|u\|_\infty \to \infty \text{ as } d \to 0,\ (d,u) \in \Gamma.$$

Suppose $(d_n, u_n) \in \Gamma,\ d_n \to d_0 \in (0, d_*],\ \|u_n\|_\infty \to \infty.$
Let $\hat{u}_n = \frac{u_n}{\|u_n\|_\infty}$. Then

$$\begin{cases} -\hat{u}_n'' = \left[g\left(e^{-k_0 x} e^{-k \int_0^x u_n(s)\,ds} \right) - d_n \right] \hat{u}_n \\ \hat{u}_n'(0) = 0,\ \hat{u}_n'(1) = 0 \end{cases}$$

Then $\{\hat{u}_n\}$ and $\{\hat{u}_n''\}$ are bounded sequences in $L^\infty[0,1]$. By standard L^p theory of elliptic equation, $\{\hat{u}_n\}$ is bounded in $W^{2,p}[0,1]$ for any $p > 1$.
By Sobolev imbedding, $W^{2,p} \underset{\text{compact}}{\hookrightarrow} C^1[0,1]$, $\{\hat{u}_n\}$ is precompact in $C^1[0,1]$. By passing to a subsequence, we assume

$$\hat{u}_n \to \hat{u} \text{ in } C^1[0,1]$$

Let

$$f_n(x) = g\left(e^{-k_0 x} e^{-k \int_0^x u_n(s)\, ds}\right).$$

$\{f_n\}$ is a bounded sequence in $L^\infty[0,1]$, by passing to a subsequence

$$f_n \to f \text{ weakly in } L^2[0,1]$$

$$0 \le f(x) \le g(1),\ 0 \le x \le 1.$$

Hence $\hat{u}(x)$ is a weak solution

$$\begin{cases} -\hat{u}''(x) = (f(x) - d_0)\hat{u}(x) \text{ in } (0,1) \\ \hat{u}'(0) = 0, \ \hat{u}'(1) = 0 \end{cases}$$

$$\hat{u} \geq 0, \ \|\hat{u}\|_\infty = 1.$$

$\because f(x) - d_0 \in L^\infty$, by strong maximal principle

$$\hat{u}(x) > 0.$$

Then

$$-\hat{u}'' + (-f(x))\hat{u} = -d_0\hat{u}$$

and

$$-d_0 = \lambda_1(-f)$$

On the other hand, $\hat{u}_n \to \hat{u} > 0$ uniformly on $[0,1]$.

$\because \|u_n\|_\infty \to \infty$

$\therefore u_n = \hat{u}\|u_n\|_\infty \to \infty$ uniformly on $[0,1]$.

$\Rightarrow e^{-k\int_0^x u_n(s)\,ds} \to 0$ uniformly on any compact subset of $(0,1]$

$\Rightarrow f_n \to 0$

$\Rightarrow f \equiv 0$

$\Rightarrow -d_0 = \lambda_1(-f) = \lambda_1(0) = 0.$

This contradicts to $d_0 \in (0, d_*]$.

$\therefore d_n \to 0$, $\|u_n\|_\infty \to \infty$, $u_n \to \infty$ uniformly on $[0,1]$.

Γ is connected $\Rightarrow$ (1) has at least one positive solution for each $d \in (0, d_*)$.

Uniqueness: Suppose for some $d \in (0, d_*)$, (1) has two positive solution u_1, u_2.
If $u_1 \leq u_2$, $u_1 \not\equiv u_2$ in $[0, 1]$, then

$$\begin{aligned} -d &= \lambda_1 \left[-g \left(e^{-k_0 x} e^{-k \int_0^x u_1(s)\, ds} \right) \right] \\ &< \lambda_1 \left[-g \left(e^{-k_0 x} e^{-k \int_0^x u_2(s)\, ds} \right) \right] = -d \quad \longrightarrow\!\longleftarrow \end{aligned}$$

Hence $u_1 - u_2$ must change sign in $(0, 1)$.
Claim: $u_1(0) \neq u_2(0)$
Let

$$\begin{aligned} v_i(x) &= \int_0^x u_i(s)\, ds \\ w_i(x) &= u_i'(x) \end{aligned}$$

Then

$$\begin{cases} v_i'(x) = u_i(x) \\ u_i'(x) = w_i(x) \\ w_i'(x) = u_i''(x) = -\left[g\left(e^{-k_0 x} e^{-v_i}\right) - d\right] u_i \\ v_i(0) = 0,\ w_i(0) = u_i'(0) = 0 \end{cases}$$

If $u_1(0) = u_2(0)$, then by uniqueness of ODE

$$(u_1(x), v_1(x), w_1(x)) \equiv (u_2(x), v_2(x), w_2(x)),\ 0 \leq x \leq 1$$

$\therefore u_1(0) \neq u_2(0)$.

Assume $u_1(0) < u_2(0)$. Then $\exists\ x_0 \in (0,1)$ s.t.

$$u_2(x) > u_1(x) \text{ in } (0, x_0)$$
$$u_1(x_0) = u_2(x_0).$$

Consider

$$\int_0^{x_0} -(u_1'' u_2)\, dx = \int_0^{x_0} \left[g\left(e^{-k_0 x} e^{-k \int_0^x u_1(s)\, ds} \right) - d \right] u_1 u_2\, dx$$
$$\int_0^{x_0} -(u_1 u_2'')\, dx = \int_0^{x_0} \left[g\left(e^{-k_0 x} e^{-k \int_0^x u_2(s)\, ds} \right) - d \right] u_1 u_2\, dx$$
$$\int_0^{x_0} -(u_1'' u_2)\, dx = -\ u_1' u_2 \Big|_0^{x_0} + \int_0^{x_0} u_1' u_2'\, dx$$
$$\int_0^{x_0} -(u_1 u_2'')\, dx = -\ u_2' u_1 \Big|_0^{x_0} + \int_0^{x_0} u_1' u_2'\, dx$$

$$
\begin{aligned}
\Rightarrow \quad & \left(u_2' u_1 - u_1' u_2\right)\Big|_0^{x_0} \\
&= \int_0^{x_0} \left[g\left(e^{-k_0 x} e^{-k \int_0^x u_1(s)\,ds} \right) - g\left(e^{-k_0 x} e^{-k \int_0^x u_2(s)\,ds} \right) \right] u_1 u_2 \, dx \\
&> 0
\end{aligned}
$$

But $u_1'(0) = u_2'(0)$, $u_1(x_0) = u_2(x_0)$, $u_1'(x_0) \geq u_2'(x_0)$,

$$
\Rightarrow \left(u_2' u_1 - u_1' u_2\right)\Big|_0^{x_0} = u_1(x_0)\left[u_2'(x_0) - u_1'(x_0)\right] \leq 0 \qquad \longrightarrow\!\longleftarrow
$$

(i) $d \mapsto u_d$ is continuous
By standard compactness arguments and uniqueness. Let $d_n \to d_0 \in (0, d_*)$. As before, then $u_{d_n} \to$ a positive solution in $C^1[0,1]$. This positive solution must be u_{d_0}.

(ii) Let $0 < d_1 < d_2 < d_*$. To show $u_{d_1}(0) > u_{d_2}(0)$.
Let $u_1 = u_{d_1}$, $u_2 = u_{d_2}$. First consider $u_1(0) < u_2(0)$.

$$0 \geq (u_1 u_2' - u_1' u_2)\Big|_0^{x_0}$$
$$= \int_0^{x_0} \left[g\left(e^{-k_0 x} e^{-k\int_0^x u_1(s)\,ds}\right) - g\left(e^{-k_0 x} e^{-k\int_0^x u_2(s)\,ds}\right)\right] u_1 u_2\, dx$$
$$+ (d_2 - d_1)\int_0^{x_0} u_1 u_2\, dx > 0 \quad \longrightarrow\!\longleftarrow$$

If $u_1(0) = u_2(0)$, then $d_2 > d_1 \Rightarrow u_2''(0) > u_1''(0)$.

$\because u_1'(0) = u_2'(0) = 0$

$\Rightarrow u_2(x) > u_1(x)$ for x small

$\Rightarrow$ we still can find interval $(0, x_0)$ as above.

(iii) Apply earlier argument,

$$(d_n, u_n) \in \Gamma, \ d_n \to 0, \ \|u_n\|_\infty \to \infty.$$

(iv) Claim:

$$-u_d''(0) = [g(1) - d]u_d(0) > 0.$$

If not, then $g(1) - d \leq 0$,

$\Rightarrow -u_d''(x) > 0 \ \forall \ 0 \leq x \leq 1$.

$\Rightarrow u_d'(1) < 0 \longrightarrow\longleftarrow (\because u_d'(1) = 0)$

$\therefore g(1) - d > 0$.

Claim: $\exists \ x_0 \in (0,1)$ such that $u_d''(x_0) = 0$.
If not, $u_d''(x) < 0 \ \forall \ x \in [0,1]$.
Then $u_d'(x)$ is strictly decreasing.
$\because u_d'(0) = 0 \therefore u_d'(1) < 0 \longrightarrow\longleftarrow$.

Claim: $u_d''(x) > 0,\ x_0 < x < 1.$

$$
\begin{aligned}
-u_d''(x) &= \left[g \left(e^{-k_0 x} e^{-k \int_0^x u_d(s)\, ds} \right) \right] u_d(x) \\
&= \left[g \left(e^{-k_0 x} e^{-k \int_0^x u_d(s)\, ds} \right) - g \left(e^{-k_0 x} e^{-k \int_0^x u_d(s)\, ds} \right) \right] \\
&< 0
\end{aligned}
$$

$\therefore\ u''(x) > 0\ \forall\ x_0 < x < 1.$

Boundedness of Positive Solutions

For simplicity we consider two populations case:

$$
\begin{aligned}
&(u_1)_t = D_1(u_1)_{xx} + [g_1(I(x,t)) - d_1]u_1,\\
&(u_2)_t = D_2(u_2)_{xx} + [g_2(I(x,t)) - d_2]u_2,\\
&(u_i)_x(0,t) = (u_i)_x(1,t) = 0, t > 0\\
&u_i(x,0) = u_{i0}(x) \geq 0, i = 1,2,\\
&\text{where } g_i(I) = \frac{m_i I}{a_i + I}, a_i > 0, m_i > 0\\
&I(x,t) = I_0 e^{-k_0 x} \exp\left(-\int_0^x \left(k_1 u_1(y,t) + k_2 u_2(y,t)\right)\, dy \right).\\
&I_0, k_0, k_1, k_2 > 0.
\end{aligned}
$$

Theorem 2

There exists $C > 0$ such that $u_1(x,t) + u_2(x,t) \le C$ for all $x \in [0,1]$ and $t > 0$.

Proof.

$$g_i(I(x,t)) \le \frac{m_i}{a_i} I(x,t) \le \sigma_i \exp\left(-k_i \int_0^x u_i \, dy\right), \; \sigma_i = \frac{m_i}{a_i} I_0.$$

Drop i, let $u = u_i$, $\sigma = \sigma_i$, $D = D_i$, $k_i = k$.

$$u_t \le D u_{xx} + \left[\sigma \exp\left(-k \int_0^x u \, dy\right) - d\right] u.$$

Integrating from $x = 0$ to $x = 1$ yields

$$\left[\int_0^1 u \, dx\right]_t \le \sigma \int_0^1 \exp\left(-k \int_0^x u \, dy\right) u \, dx - d \int_0^1 u \, dx.$$

Let

$$w(t) = \int_0^1 u(x,t)\,dx,\ v(x,t) = \int_0^x u(y,t)\,dy.$$

Then

$$\int_0^1 \exp\left(-k\int_0^x u\,dy\right) u\,dx = \int_0^1 e^{-kv} v_x\,dx$$
$$= \int_{v(0,t)}^{v(1,t)} e^{-kv}\,dv = k^{-1}\left[e^{-kv(0,t)} - e^{-kv(1,t)}\right]$$
$$= k^{-1}\left[1 - e^{-kw(t)}\right]$$

Therefore

$$
\begin{aligned}
w_t &\le \sigma k^{-1}\left(1 - e^{-kw}\right) - dw \\
w_t + dw &\le C_0 := \sigma k^{-1} \\
w(t) &\le C := w(0) + \frac{C_0}{d}. - - - - - - (**)
\end{aligned}
$$

Set

$$W(t) = \max_{x\in[0,1],\, s\in[0,t]} u(x,s).$$

$W(t)$ is non-decreasing in t. Suppose $W(t) \to \infty$ as $t \to \infty$.

We want to get a contradiction.
Find $\{t_n\} \uparrow \infty$ s.t. $W(t_n) = \max_{x\in[0,1]} u(x, t_n)$.
Define $v_n(x, t) = \frac{u(x,t+t_n-1)}{W(t_n)}$.
Then

$$\begin{aligned}
&(v_n)_t = D(v_n)_{xx} + c_n v_n, \\
&(v_n)_x = 0 \quad \text{for} \quad x \in \{0, 1\}, t > 0, \\
&v_n(x, 0) \in [0, 1],
\end{aligned}$$

where

$$c_n(x, t) = g(I(x, t + t_n - 1)) - d$$

and $\exists M > 0$ s.t. $|c_n| \leq M = \sigma + d$.

By comparison we have

$$0 \le v_n(x,t) \le e^{Mt},\ x \in [0,1],\ t \ge 0.$$

Apply standard parabolic regularity, $\{v_n\}$ is bounded in $C^{1+\alpha,\alpha}\left([0,1] \times [\frac{1}{2},2]\right)$, $\alpha \in (0,1)$.
By passing to a subsequence, assume

$$v_n \to v^* \text{ in } C^{1,0}\left([0,1] \times [\frac{1}{2},2]\right).$$

Since $|c_n| \le M$.
We assume $c_n \to c$ weakly in $L^2\left([0,1] \times [\frac{1}{2},2]\right)$.

Then v^* is a weak solution of

$$v_t^* = Dv_{xx}^* + cv^*, \; x \in [0,1], \; t \in [\frac{1}{2}, 2],$$
$$v_x^*(0,t) = v_x^*(1,t) = 0, \; t \in [\frac{1}{2}, 2]$$
$$v^*(x,t) \in [0, e^{2M}], \; x \in [0,1], \; t \in [\frac{1}{2}, 2].$$

Since $\max_{x\in[0,1]} v_n(x,1) = 1$ then $\max_{x\in[0,1]} v^*(x,1) = 1$.
By strong maximum principle,

$$v^*(x,1) \geq \delta_0 > 0.$$

Hence $v_n(x,1) \geq \frac{\delta_0}{2}$ for n large.

$$u(x,t_n) \geq \left(\frac{\delta_0}{2}\right) W(t_n).$$

$$w(t_n) = \int_0^1 u(x,t_n)\,dx \geq \left(\frac{\delta_0}{2}\right) W(t_n) \to \infty.$$

This contradicts to $(**)$.
$\therefore u(x,t) \leq C_1$ for some C_1, $\forall t > 0$.

□

Now we consider single population case

$$\begin{aligned}
&u_t = Du_{xx} + (g(I(x,t)) - d)u \\
&u_x(0,t) = u_x(1,t) = 0. \\
&u(x,0) = u_0(x) \geq, \neq 0 \\
&I(x,t) = I_0 e^{-k_0 x} \exp\left(-\int_0^x ku(y,t)\,dy\right).
\end{aligned}$$

Theorem 3 [H. Ishii and I. Takagi, JMB 1982]

Lemma 1:
Let $u_0,\ \widetilde{u_0} \in C[0,1]$, $0 \le u_0 < \widetilde{u_0}$.
Let $u(x,t), \widetilde{u}(x,t)$ be solutions with $u(x,0) = u_0$, $\widetilde{u}(x,0) = \widetilde{u_0}$.
Then $v(x,t) < \widetilde{v}(x,t)$, $t > 0$, $x \in [0,1]$,
where $v(x,t) = \int_0^x u(y,t)\,dy$, $\quad \widetilde{v}(x,t) = \int_0^x \widetilde{u}(y,t)\,dy$.

Theorem 3:
$u(x,t) \to u^*(x)$ as $t \to \infty$ uniformly in x.
Proof.

Step1: Let $\underline{u}_0 = \varepsilon\phi < u_0$, $\underline{u}(x,0) = \underline{u}_0(x)$.
Then $\underline{u}_t(x,t) \ge \delta_0 > 0$, t is small.
$\underline{u}(x,t_1) < \underline{u}(x,t_2)$, $0 < t_1 < t_2 \le \delta$
Let $u_0^1(x) = \underline{u}(x,t_1)$, $u_0^2(x) = \underline{u}(x,t_2)$,
$\underline{u}^1(x,t) = \underline{u}(x,t_1+t)$, $\underline{u}^2(x,t) = \underline{u}(x,t_2+t)$.

By Lemma 1,

$$\underline{v}^1(x,t) < \underline{v}^2(x,t),\ \forall t > 0,\ x \in [0,1],$$
$$\underline{v}(x,t+t_1) < \underline{v}(x,t+t_2),\ x \in [0,1],\ t > 0.$$
$$\text{where } \underline{v}(x,t) = \int_0^x \underline{u}(s,t)\,ds$$

$\underline{v}(x,t)$ is $\uparrow$ in t.
By Theorem 2, $\underline{v}(x,t) \leq M$.
By parabolic regularity,

$$\underline{v}(\cdot,t) \to v_*(\cdot) \text{ in } C^1([0,1]) \text{ as } t \to \infty.$$
$$\underline{u}(\cdot,t) \to v_*'(\cdot) \text{ in } C[0,1].$$

By parabolic regularity, $\underline{u}(\cdot,t) \to v_*'(\cdot)$ in $C^1[0,1]$.

$v_*'(\cdot)$ must be a steady state of (1).
$v_*(0) = 0,\ v_*(x) > 0 \implies v_*'(x) = u^*(x),\ 0 \leq x \leq 1.$
(by uniqueness of steady state)
Let $M > 0,\ -d_M = \lambda_1 \left(-g \left(e^{-(k_0+M)x}\right)\right)$.
$d_M \to 0$ as $M \to \infty$.
Choose $M > 0$ large s.t. $0 < d_M < d$.
Let $\psi(x)$ be the positive eigenfunction corresponding to

$$-d_M = \lambda_1 \left(-g \left(e^{-(k_0+M)x}\right)\right).$$

Choose $\varepsilon > 0$ small s.t. $\varepsilon^{-1}\psi > u_0$.
Let $\overline{u}_0 = \varepsilon^{-1}\psi,\ \overline{u}(x,0) = \varepsilon^{-1}\psi = \overline{u}_0$.
Then $\overline{u}_t \leq -\delta < 0$.
$\overline{u}(x,t_1) > \overline{u}(x,t_2),\ 0 < t_1 < t_2$ small.

Then $\overline{v}(x,t) = \int_0^x \overline{u}(s,t)\,ds$ is $\downarrow$ in t.(Lemma 1)

Choose $\varepsilon > 0$ s.t. $\varepsilon^{-1}\psi > u_0$, $\varepsilon^{-1}\psi > u^*$.

$$\lim_{t\to\infty} \overline{v}(x,t) = \overline{v}^*(x) \geq v^*(x) = \int_0^x u^*(s)\,ds.\text{(Lemma 1)}$$

$\overline{v}(\cdot,t) \to \overline{v}^*(\cdot)$ in $C^1[0,1]$. (parabolic regularity)

$\overline{u}(\cdot,t) \to \overline{v}^{*'}(\cdot)$ in $C[0,1]$ and $C^1[0,1]$. (parabolic regularity).

$\because\ \overline{v}^{*'}(x) \equiv u^*(x)$

$\varepsilon\phi \leq u_0 \leq \varepsilon^{-1}\psi$

$\overline{u}(\cdot,t) \to u^*$ as $t \to \infty$.

$$\begin{aligned} \text{Lemma 1} \quad &\Longrightarrow \underline{v}(x,t) \leq v(x,t) = \textstyle\int_0^x u(s,t)\,ds \leq \overline{v}(x,t) \\ &\Longrightarrow v(x,t) \to v^*(x) = \textstyle\int_0^x u^*(s)\,ds \\ &\Longrightarrow u(x,t) = v'(x,t) \to (v^*)'(x) = u^*(x) \end{aligned}$$

$\square$

Extinction Results: Theorem 4

Let $D_1 = D_2 = d$.
If $(g_2(I) - d_2) - (g_1(I) - d_1) < -\delta < 0$ for all $I \geq 0$, then $v(x,t) \to 0$ as $t \to \infty$.

Proof. Let

$$w(x,t) = \frac{v(x,t)}{u(x,t)}.$$

Then

$$\begin{aligned}\frac{\partial w}{\partial t} =& d\frac{\partial^2 w}{\partial x^2} + \frac{\partial w}{\partial x}\left(\frac{2d}{u(x,t)}\frac{\partial u}{\partial x}\right) \\ &+ w\left[(g_2(I) - d_2) - (g_1(I) - d_1)\right] \\ &\frac{\partial w}{\partial x}(0,t) = 0, \frac{\partial w}{\partial x}(1,t) = 0.\end{aligned}$$

Let $w(x,t) = z(x,t)e^{-\beta t}$, $\beta > 0$ to be determined,
$c(x,t) = (g_2(I(x,t)) - d_2) - (g_1(I(x,t)) - d_1) \le -\delta < 0$.
Then

$$
\begin{aligned}
& d\frac{\partial^2 z}{\partial x^2} - \frac{\partial z}{\partial t} + \frac{2d}{u(x,t)}\frac{\partial u}{\partial x}\frac{\partial z}{\partial x} + (\beta + c(x,t))z = 0, \\
& \frac{\partial z}{\partial x}(0,t) = 0, \ \frac{\partial z}{\partial x}(1,t) = 0.
\end{aligned}
$$

Let $0 < \beta < \delta$. Then $\beta + c(x,t) < 0$ for $t \ge t_0$, $0 < x < 1$.
From maximum principle, it follows that

$$z(x,t) \le z(x,t_0), \ t \ge t_0, \ 0 \le x \le 1.$$

$$v(x,t) \le u(x,t)z(x,t_0)e^{-\beta(t-t_0)} \le Me^{-\beta t}.$$

□

Uniformly Persistence Results

Theorem (HW)

Suppose $T(t)$ be a C^0-semigroup on X satisfies $T(t) : X^0 \to X^0$, $T(t) : \partial X^0 \to \partial X^0$, $X = X^0 \cup \partial X^0$, X^0 is an open set.
Assume $T(t)$ satisfies

(i) *There exists $t_0 \geq 0$ such that $T(t)$ is compact for $t > t_0$.*

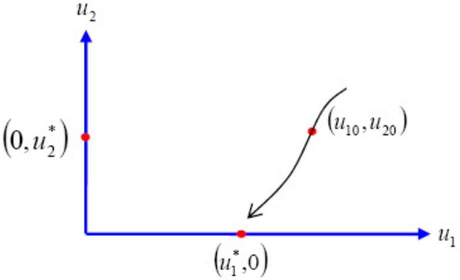
(ii) *$T(t)$ is point dissipative in X.*

(iii) *$\widetilde{A_\partial}$ is isolated and has an acyclic covering M.*

Then $T(t)$ is uniformly persistent $\Leftrightarrow$ for each $M_i \in M$, $W^s(M_i) \cap X^0 = \phi$.

Let $X = C^+[0,1] \times C^+[0,1]$.
Let $u_i^*(x)$ be the unique steady state of

$$(u_i)_t = D_i (u_i)_{xx} + (g_i(I(x,t)) - d_i)\, u_i$$
$$(u_i)_x(0,t) = 0,\ (u_i)_x(1,t) = 0,\ i = 1,2.$$
$$I(x,t) = I_0 e^{-k_0 x} \exp\left(-\int_0^x k_i u_i(s,t)\, ds\right)$$



To apply [HW] uniformly persistence theorem. We need to show $W^s(u_1^*, 0) \cap X^0 = \phi$ and $W^s(0, u_2^*) \cap X^0 = \phi$.

If $W^s(u_1^*, 0) \cap X^0 \neq \phi$, then $\exists\,(u_{10}, u_{20})$ s.t.

$$u_1(x, t; u_{10}, u_{20}) \to u_1^*(x)$$
$$u_2(x, t; u_{10}, u_{20}) \to 0$$

as $t \to \infty$.
Then

$$I(x, t) \to I_0 e^{-k_0 x} \exp\left(-k_1 \int_0^x u_1^*(s)\, ds\right).$$

Consider

$$(2) \quad \frac{\partial u_2}{\partial t} = D_2(u_2)_{xx} + \left(g_2 \left(I_0 e^{-k_0 x} \exp\left(-k_1 \int_0^x u_1^*(s)\, ds \right) \right) - d_2 \right) u_2.$$

If $0 < d_2 < (d_2)_*$, where

$$-(d_2)_* = \lambda_1 \left(-g_2 \left(I_0 e^{-k_0 x} \exp\left(-k_1 \int_0^x u_1^*(s)\, ds \right) \right) \right),$$

then $u_2(x,t) \to \widetilde{u_2}(x) > 0$ where $\widetilde{u_2}(x)$ is the unique positive steady state of (2), we obtain a contradiction.
$\therefore W^s(u_1^*, 0) \cap X^0 = \phi$.

Theorem 5

Similarly, if $0 < d_1 < (d_1)_*$, where

$$-(d_1)_* = \lambda_1 \left(-g_1 \left(I_0 e^{-k_0 x} \exp \left(-k_1 \int_0^x u_2^*(s)\, ds \right) \right) \right),$$

then $W^s(0, u_2^*) \cap X^0 = \phi$.
Thus we obtain the following uniformly persistent result:

Theorem 5
If $0 < d_1 < (d_1)_*$ and $0 < d_2 < (d_2)_*$, then the system (1) is uniformly persistent and there exists a positive steady state of (1).