Motion by Curvature of Planar Curves with Two Free End Points

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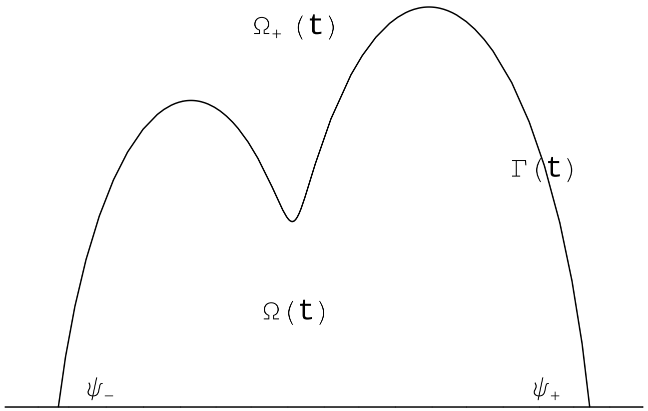
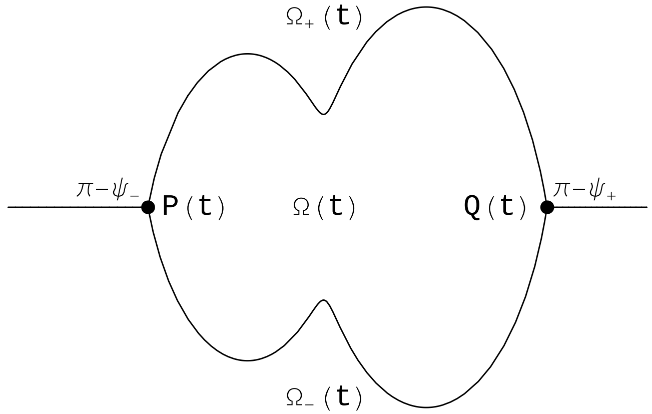
In this talk, we are interested in the following problem:

Problem (P): Given an initial curve $\Gamma(0)$, find a family of curves $\{\Gamma(t)\}_{0<t<T}$ that lie on the upper-half plane, have end points on the x-axis with contact angle ψ_- on the left and ψ_+ on the right, and evolve according to the motion by curvature.

- Motion by curvature: $V(\text{normal velocity}) = \kappa(\text{curvature})$.
- This problem arises in the study of the evolution of three grain domains in polycrystals.



(a)



(b)

Figure (b) is a schematic snap shot of a diminishing grain domain $\Omega(t)$ surrounded by two other grain domains $\Omega_+(t)$ and $\Omega_-(t)$; the dots $P(t)$ and $Q(t)$ are the triple junctions of these three grain domains. When $\Omega(t)$ is symmetric about the x-axis, figure (a), modelled by problem (P), is the upper-half part of figure (b).

- A grain domain is a periodic lattice structure of composite particles of a crystal.
- A grain boundary is the intersection of two grain domains at which orientations of different lattices do not match.
- Grain boundaries are often modelled by the curvature flows.
- A triple junction is the meeting place of three grain domains.
- In general, it is assumed that the intersection angles (φ) at a triple junction and the three interfacial energy densities (σ) satisfy the Herring condition.

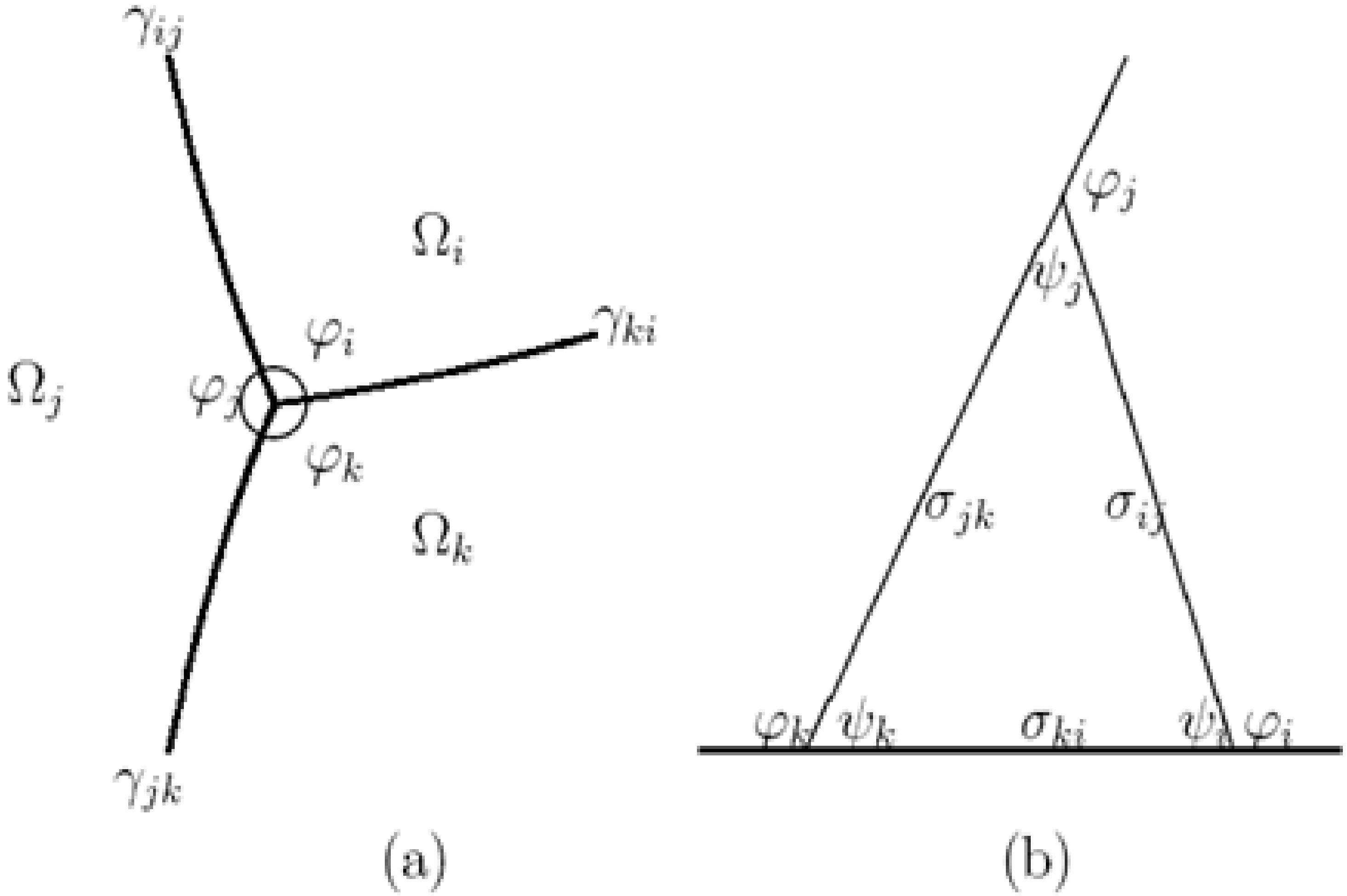


FIGURE 1

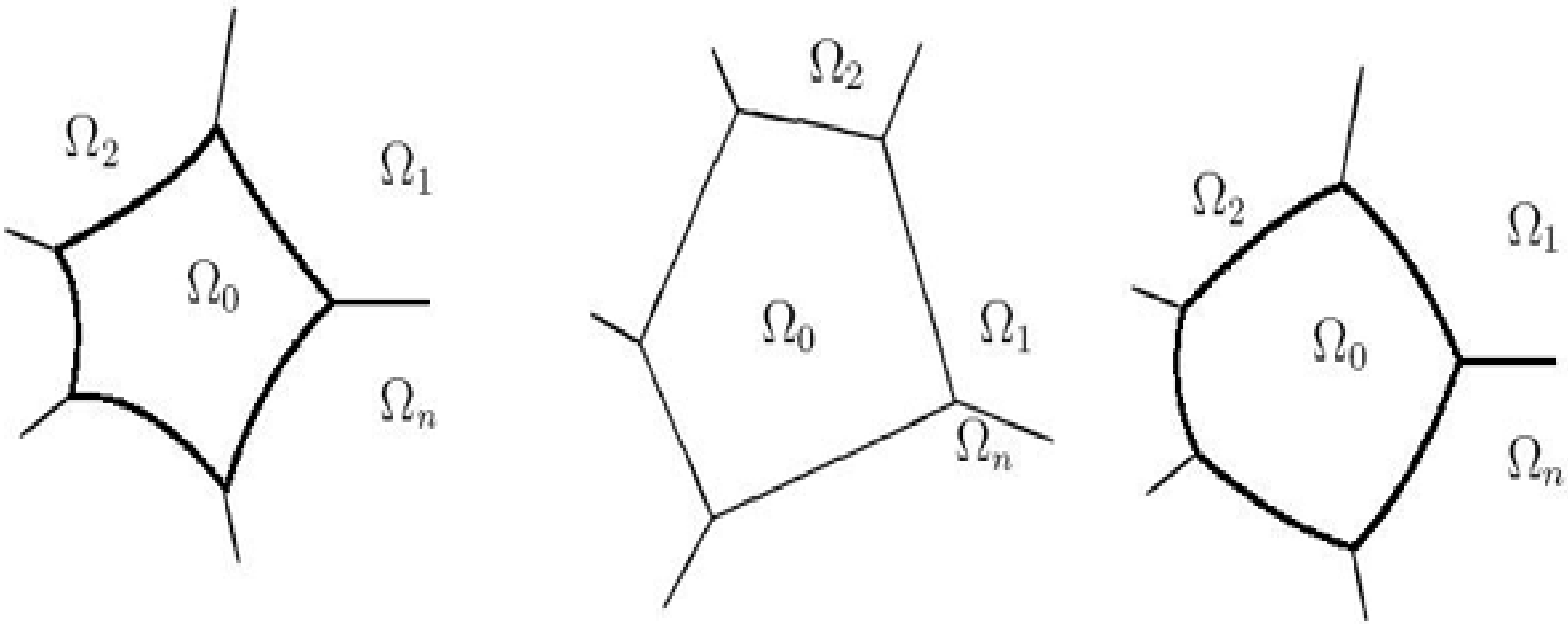


FIGURE 2. Left: self-similar expanding; middle: stationary; right: self-similar shrinking.

– a movie from David Kinderlehrer

PDE Formulations of (P):

1. Motion of Particles: We can regard $\Gamma(0)$ as the union of the positions of a collection of particles so that

$\Gamma(0) = \{(x^0(z), y^0(z)) \mid 0 \leqslant z \leqslant 1\}$ with

$|x_z^0(z)| + |y_z^0(z)| > 0$ for all $z \in [0,1]$,

$y^0(0) = y^0(1) = 0$,

$x_z^0(0) = y_z^0(0) \cot \psi_-$,

$x_z^0(1) = -y_z^0(1) \cot \psi_+$.

Let

$$\Gamma(t) := \{(x(z,t), y(z,t)) \mid z \in [0,1]\}.$$

(P): Find $X = (x, y)$ such that

$$\begin{cases} x_t = \dfrac{x_{zz}}{x_z^2 + y_z^2},\ y_t = \dfrac{y_{zz}}{x_z^2 + y_z^2},\ z \in (0,1), t \in (0,T), \\ y(0,t) = 0,\ y(1,t) = 0,\ t \in [0,T), \\ x_z(0,t) = y_z(0,t) \cot \psi_-,\ t \in [0,T), \\ x_z(1,t) = -y_z(1,t) \cot \psi_+,\ t \in [0,T), \\ x(z,0) = x^0(z),\ y(z,0) = y^0(z),\ z \in [0,1]. \end{cases} \tag{1}$$

- This is the most general formulation.

2. Polar Coordinates Formulation:

Fix a reference point $x_0 + 0\mathbf{i} \in \mathbb{C} = \mathbb{R}^2$, set

$$\Gamma(t) = \{x_0 + R(\varsigma, t)e^{\mathbf{i}\varsigma} \mid 0 \leqslant \varsigma \leqslant \pi\}.$$

Then problem (P) can be expressed as

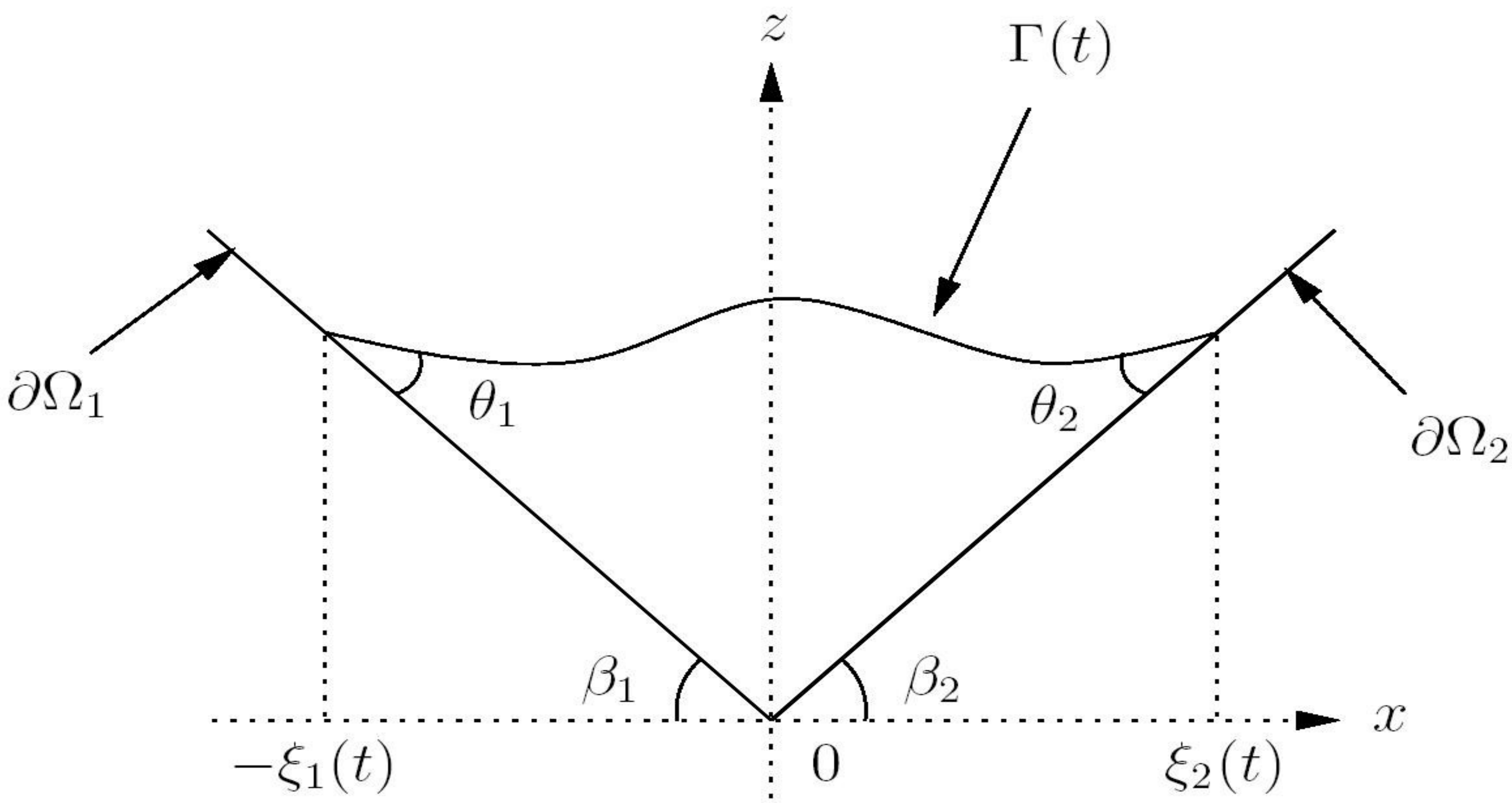
$$\begin{cases} R_t = \dfrac{RR_{\varsigma\varsigma} - 2R_\varsigma^2 - R^2}{R(R^2 + R_\varsigma^2)}, \ \varsigma \in (0, \pi), t \in (0, T), \\ R_\varsigma(0, t) = -R(0, t) \cot \psi_+, \ \ t \in [0, T) \\ R_\varsigma(\pi, t) = R(\pi, t) \cot \psi_-, \ \ t \in [0, T), \\ R(\varsigma, 0) = R^0(\varsigma), \varsigma \in [0, \pi]. \end{cases} \tag{2}$$

3. Evolution of A Graph: When $\psi_\pm \in (0, \pi/2)$ and $\Gamma(0)$ is a graph $y = u^0(x)$, $x \in [l_-^0, l_+^0]$, one can expect that $\Gamma(t)$ is also a graph given by $y = u(x,t)$, $x \in [l_-(t), l_+(t)]$.

(P): find unknowns u and $\{l_\pm(t)\}$ such that

$$\begin{cases} u_t = (\arctan u_x)_x, \ x \in (l_-(t), l_+(t)), \ t \in (0,T), \\ u(l_\pm(t), t) = 0, \ t \in [0,T), \\ u_x(l_\pm(t), t) = \mp \tan \psi_\pm, \ t \in [0,T), \\ u(x,0) = u^0(x), \ x \in [l_-(0), l_+(0)] := [l_-^0, l_+^0]. \end{cases} \quad (3)$$

¶ It is a free boundary problem for a scalar equation.



z
$\Gamma(t)$
$\partial\Omega_1$
θ_1
θ_2
$\partial\Omega_2$
β_1
β_2
x
$-\xi_1(t)$
0
$\xi_2(t)$

Set $\psi_1 := \theta_1 - \beta_1, \psi_2 := \beta_2 - \theta_2$.

¶ Chang-G.-Kohsaka (AA, 2003) - expanding case ($\psi_1 < \psi_2$).

¶ G.-Hu (QAM, 2006) - area-preserving ($\psi_1 = \psi_2$) and shrinking ($\psi_1 > \psi_2$) cases.

• Study the free boundary formulation in a conical domain with open angle, i.e., $\pi - (\beta_1 + \beta_2)$, strictly less than π.

• Global or non-global existence and uniqueness of solutions are established. Moreover, the asymptotic behaviors, as $t \to T^-, T \leq \infty$, are also studied.

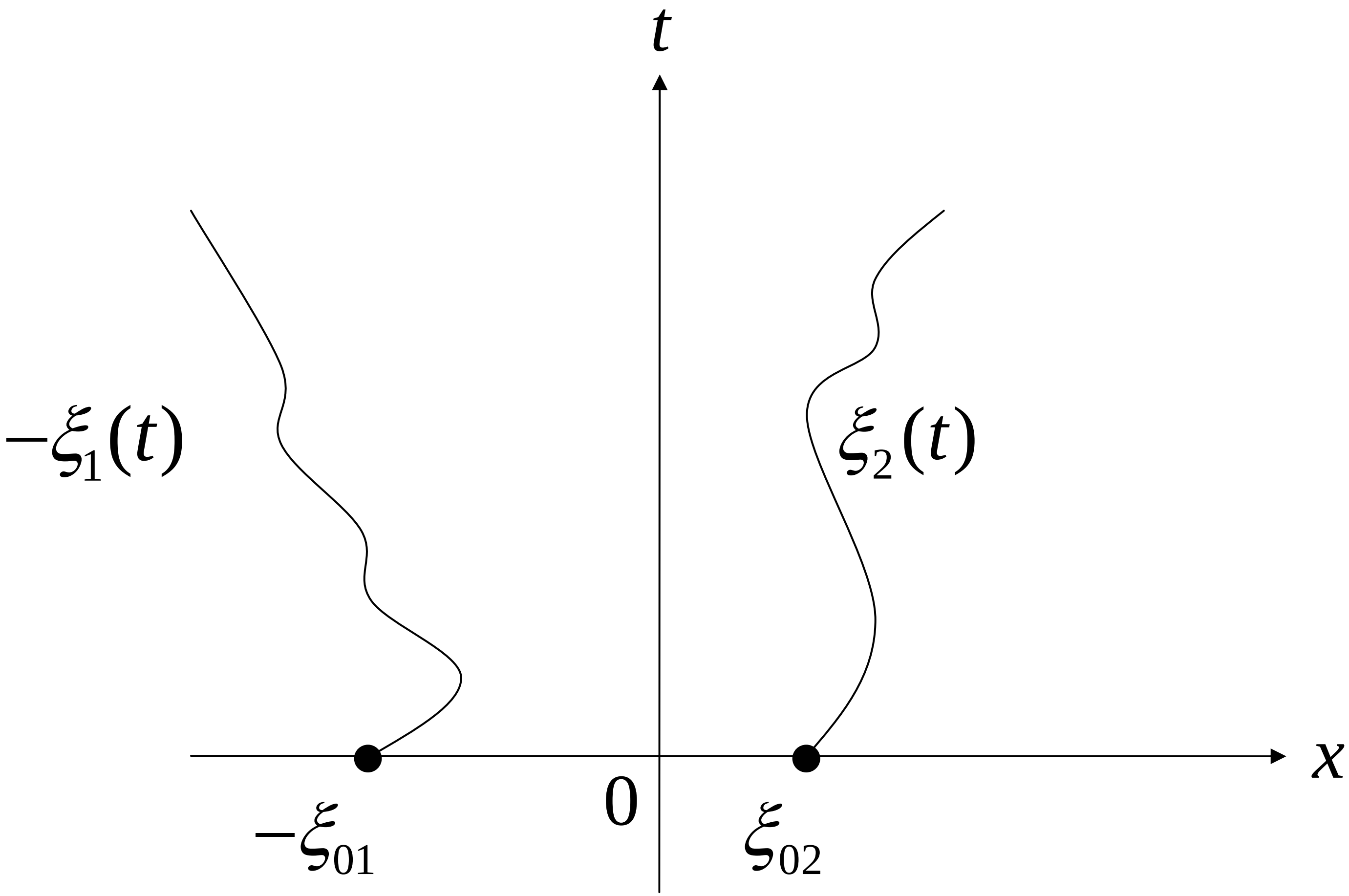
¶ Only in the shrinking case, the open angle can be equal to π.

Some Difficulties of Problem (P):

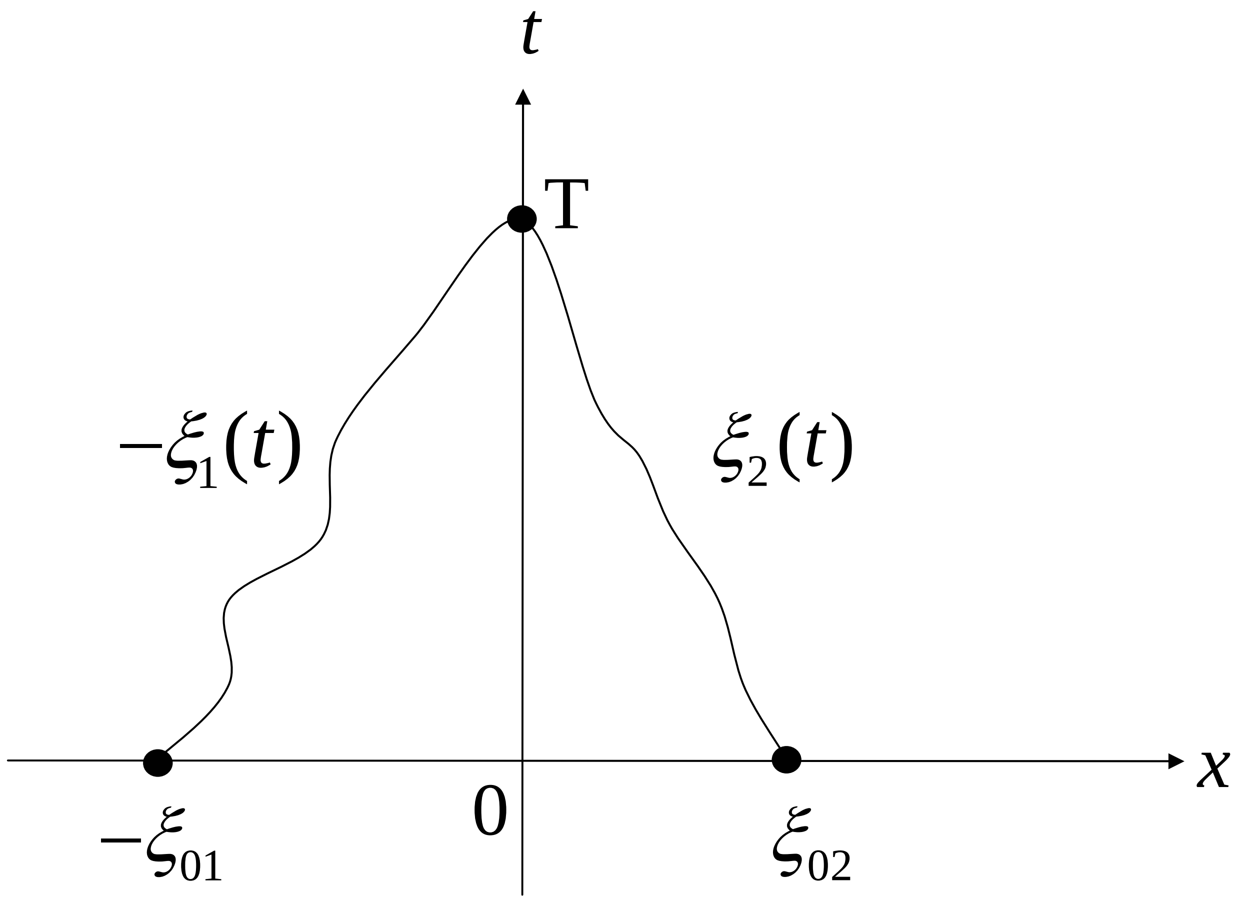
• Need to eliminate Grim Reaper type singularity, or, needle-type singularity of solution.

• Existence and uniqueness of self-similar shrinking solution is highly nontrivial.

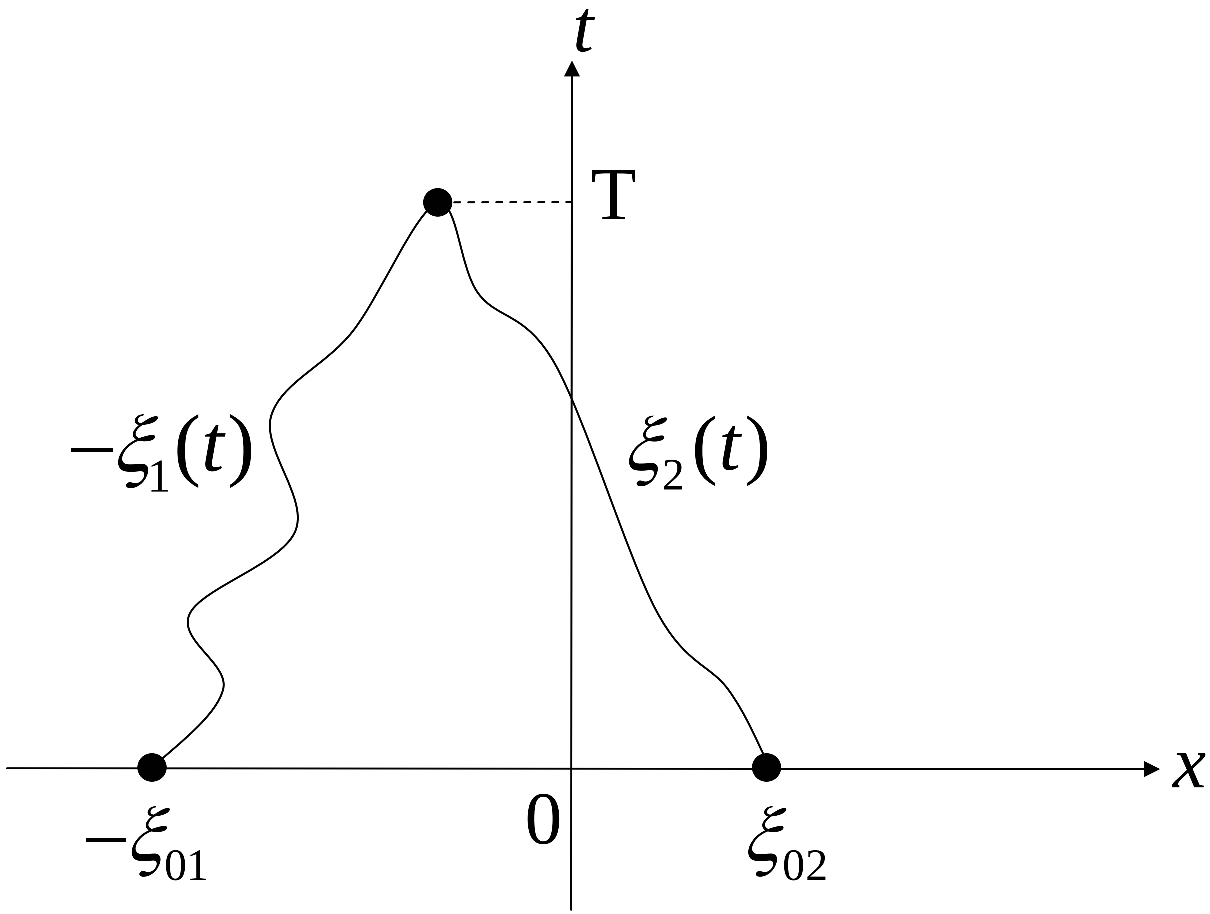
• Since the open angle is π, the center in polar coordinates might need to be changed after some time – Goto domain3



t
$-\xi_1(t)$
$\xi_2(t)$
x
0
$-\xi_{01}$
ξ_{02}



t
T
$-\xi_1(t)$
$\xi_2(t)$
x
$-\xi_{01}$
0
ξ_{02}


t
T
$-\xi_1(t)$
$\xi_2(t)$
x
$-\xi_{01}$
0
ξ_{02}

4. Velocity Formulation: Write

$$X_z(z,t) = v(z,t)e^{\mathbf{i}\psi(z,t)}.$$

$$\begin{cases} v_t = (v^{-2}v_z)_z - v^{-1}\psi_z^2, \ z \in (0,1), t \in (0,T), \\ \psi_t = v^{-2}\psi_{zz}, \ z \in (0,1), t \in (0,T), \\ v_z(0,t) = -v(0,t)\psi_z(0,t)\cot\psi_-, \ t \in [0,T), \\ v_z(1,t) = v(1,t)\psi_z(1,t)\cot\psi_+, \ t \in [0,T), \\ \psi(0,t) = \psi_-, \ \psi(1,t) = -\psi_+, \ t \in [0,T), \\ v(z,0) = v^0(z), \ \psi(z,0) = \psi^0(z), \ z \in [0,1]. \end{cases}$$

5. Curvature Formulation:

$$\begin{cases} \kappa_t = \kappa^2(\kappa_{\theta\theta} + \kappa), \ \theta \in (-\psi_-, \psi_+), \ t \in (0, T), \\ \kappa_\theta(\theta, t) \sin\theta = \kappa(\theta, t) \cos\theta, \ \theta = \pm\psi_\pm, \ \ t \in [0, T), \\ \kappa(\theta, 0) = \kappa^0(\theta), \ \theta \in [-\psi_-, \psi_+]. \end{cases}$$

- Friedman-McLeod (ARMA, 1986): $\kappa(\pm\psi_\pm, t) = 0$
- Angenent (JDG, 1991): replacing $[-\psi_-, \psi_+]$ by $\mathbb{R}$ and using $2n\pi$ (n positive integer) periodic boundary conditions
- see also Grayson (JDG, 1987): evolution of planar curves

Existence and Uniqueness of Solution:

Theorem 1 *Let* $\psi_+, \psi_- \in (0, \pi)$ *and assume that* $\Gamma(0) \in C^{1+\alpha}$ *for some* $\alpha \in (0,1)$. *Then there exists a positive* T *such that* (1) *admits a unique solution*

$$(x, y) \in C^\infty([0,1] \times (0,T)) \cap C^{1+\alpha,(1+\alpha)/2}([0,1] \times [0,T)),$$

and T *is the time of blow-up of curvature:*

$$\lim_{t \nearrow T} \|\kappa\|_{L^\infty(\Gamma(t))} = \infty.$$

- Apply a fixed-point argument to the particle formulation.
- The sup-norm of κ does not depend on any parameterization of $\Gamma(t)$, since the curvature is a geometric quantity.

Geometric Properties:

Theorem 2 *Assume that* $\psi_\pm > 0$, $\psi_+ + \psi_- \leqslant \pi$, *and* $\Gamma(0)$ *is a simple curve whose interior lies in the upper-half plane. Then for each* $t \in (0, T)$, $x(0,t) < x(1,t)$ *and* $\Gamma(t)$ *is also a simple curve with interior lying in the upper-half plane. In addition, the area* $A(t)$ *of the region bounded by* $\Gamma(t)$ *and the* x*-axis is given by*

$$A(t) = A(0) - [\psi_- + \psi_+]t \quad \forall t \in [0, T),$$

so that $T \leqslant T_{\max} := A(0)/(\psi_+ + \psi_-)$.

Self-similar Solution:

Theorem 3 *Assume that* $\psi_\pm \in (0, \pi/2]$. *Then there exists a unique self-similar shrinking solution.*

Recall the formulation in polar coordinates (2) and set

$$\varphi := \text{arccot}\frac{R_\varsigma}{R}.$$

Then the problem can be written as

$$\begin{cases} RR_t = -1 - \varphi_\zeta, R_\varsigma = R\cot\varphi, \varsigma \in (0,\pi), t \in (0,T), \\ \varphi(0,t) = \pi - \psi_+, \varphi(\pi,t) = \psi_-, t \in [0,T), \\ R(\varsigma,0) = R^0(\varsigma),\ \varsigma \in [0,\pi]. \end{cases}$$

A self-similar shrinking solution can be put in the form

$$R(\varsigma, t) = \sqrt{2(T-t)\rho(\varsigma)},$$
$$\varphi(\varsigma, t) = \psi(\varsigma),\ \forall\, \varsigma \in [0, \pi], t \in [0, T).$$

This reduces to to solve the ODE system

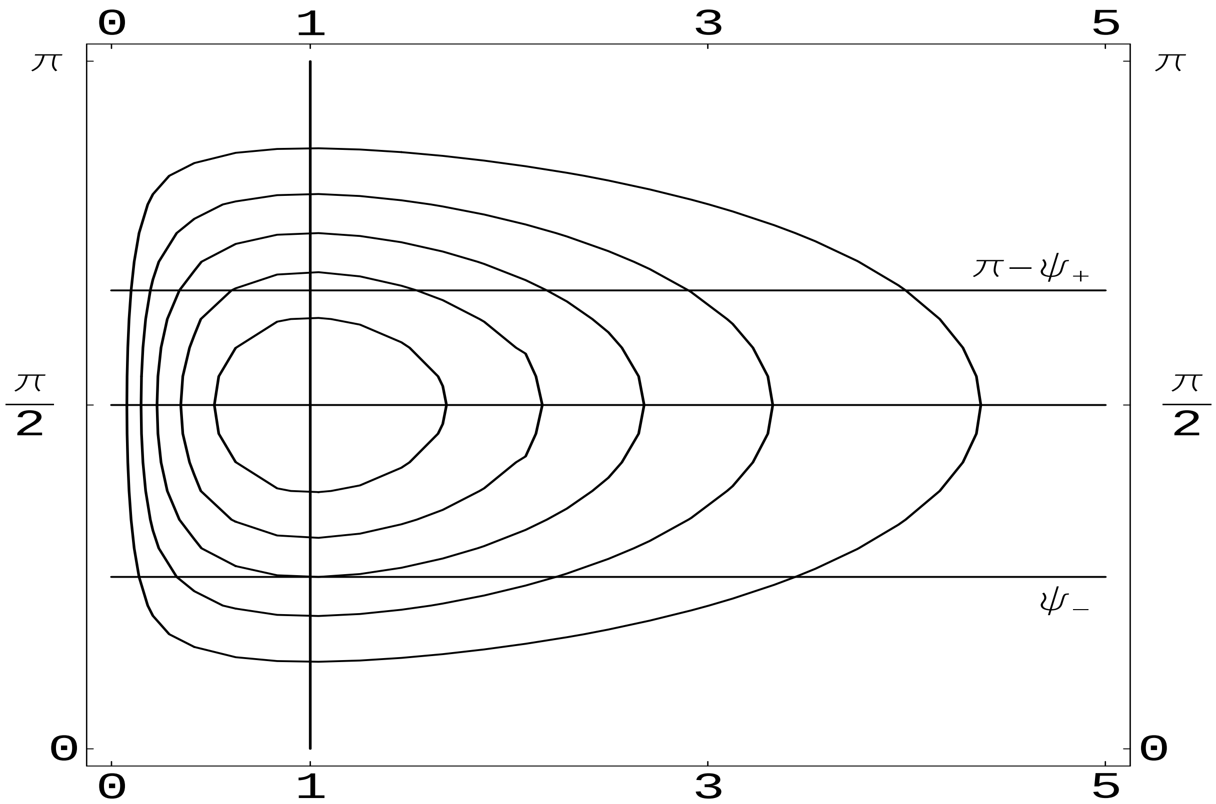
$$\rho' = 2\rho \cot \psi, \quad \rho > 0, \quad \psi' = \rho - 1 \text{ in } [0, \pi], \tag{4}$$

subject to the boundary conditions

$$\psi(0) = \pi - \psi_+, \qquad \psi(\pi) = \psi_-. \tag{5}$$

- First integral $\leftrightarrow [\ln \sin^2 \psi + \ln \rho - \rho]' = 0$
- A generic trajectory to (4) is given by $e^{\rho - 1}/\rho = c \sin^2 \psi$

for some constant $c \geq 1 \leftrightarrow \gamma(c)$ (counterclockwise)



- Let $B_1(c,\varphi)$ ($B_2(c,\varphi)$) be the left (right) intersection point of $\psi = \varphi$ with $\gamma(c)$.
- Let $\ell_1(c,\varphi)$ ($\ell_2(c,\varphi)$) be the "time" spent on $\gamma(c)$ from the leftmost point of $\gamma(c)$ to $B_1(c,\varphi)$ ($B_2(c,\varphi)$).

¶ Key Idea: To evaluate the "time" spent on the trajectory $\gamma(c)$ so that $\ell_i(c,\psi_+) + \ell_j(c,\psi_-) = \pi,\, i,j \in \{1,2\}$.

- For every $\varphi \in (0,\pi/2]$, we have $\ell_2(\infty,\varphi) = \pi/2$, $d\ell_2(c,\varphi)/dc < 0, \forall\, c \gg 1$, and

$$\frac{d}{dc}\Big(c(c-1)\frac{d}{dc}\ell_2(c,\varphi)\Big) < 0,\ \ \forall\, c > \frac{1}{\sin^2\varphi}.$$

- Let $\omega(c)$ be the whole period of $\gamma(c)$. Then

$$\omega(c) = 2\ell_2(c, \pi/2).$$

- Let $\omega_1(c)$ be the "time" spent on $\gamma(c)$ from the leftmost point to the bottom point. Then we have

$$\lim_{c \searrow 1} \omega(c) = \sqrt{2}\,\pi, \ \lim_{c \nearrow \infty} \omega(c) = \pi,$$

$$\lim_{c \searrow 1} \omega_1(c) = \sqrt{2}\,\pi/4, \ \lim_{c \nearrow \infty} \omega_1(c) = \pi/2,$$

$$\omega'(c) < 0, \ \omega_1'(c) > 0, \ \forall\, c \in (1, \infty).$$

Theorem 4 (Asymptotic Behavior) *Assume that* $\Gamma(0)$ *is a graph and* $0 < \psi_\pm < \pi/2$ *such that* u^0 *satisfies*

$$u^0 \in C^\infty([l^0_-, l^0_+]),\ u^0(l^0_\pm) = 0,$$
$$u(\cdot) > 0 \text{ in } (l^0_-, l^0_+),\ \mp u^0_x(l^0_\pm) = \gamma_\pm > 0.$$

Then (3) *admits a unique solution with* $T = T_{\max}$*, and as* $t \nearrow T$*,* $\Gamma(t)$ *shrinks to a point in a self-similar manner.*

• Using the standard blow-up technique in parabolic problem, we can derive the convergence to the self-similar shrinking solution as $t \nearrow T$.

We make the change of dependent and independent variables:

$$z = \frac{x}{\sqrt{2(T-t)}}, \qquad s = -\ln\sqrt{2(T-t)},$$

$$U(z,s) := u(x,t)/\sqrt{2(T-t)},$$

$$L_\pm(s) = l_\pm(t)/\sqrt{2(T-t)}.$$

Set $s_0 = -\ln\sqrt{2T}$. Then the functions $(U, L_\pm)$ satisfies

$$\begin{cases} U_s = [a(U_z)]_z - zU_z + U, \ z \in (L_-(s), L_+(s)), \ s > s_0, \\ U(L_\pm(s), s) = 0, \ U_z(L_\pm(s), s) = \mp\gamma_\pm, \ s > s_0, \end{cases}$$

where $a(s) = \arctan(s)$ and $\gamma_\pm = \tan\psi_\pm$.

Some Estimates and Properties:

1. For every $t \in [0, T)$ and $x \in [l_-(t), l_+(t)]$,

$$|u_x(x,t)| \leqslant M,\ u_t(x,t) \leqslant M,\ u_{xx} \leqslant M$$

for some positive constant M.

2. There exists a constant C that depends only on u^0 such that

$$u_t(x,t) \geqslant -\frac{Ch(0)}{h(t)},\ u_{xx}(x,t) \geqslant -\frac{Ch(0)}{h(t)},$$

where $h(t) := \max_{l_-(t)<x<l_+(t)} u(x,t)$.

3. There exists $t_* \in [0, T)$ and $\xi \in C^1([t_*, T))$ such that for each $t \in [t_*, T)$,

$$u_x(\cdot, t) > 0 \text{ in } [l_-(t), \xi(t)),$$

$$u_x(\xi(t), t) = 0 > u_{xx}(\xi(t), t),$$

$$u_x(\cdot, t) < 0 \text{ in } (\xi(t), l_+(t)].$$

In addition, $\dot{l}_-(t) > 0$, $\dot{l}_+(t) < 0$ for all $t \in [t_*, T)$.

4. There exists a constant M and a time $t_* \in [0, T)$ such that

$$u_{xx}(x, t) \leqslant M u(x, t), \; \forall\, x \in (l_-(t), l_+(t)), \; t \in [t_*, T).$$

5. Key Estimate: there exists a constant $C > 0$ such that

$$\sqrt{T-t} \leqslant C\ell(t) \leqslant C^2 h(t) \leqslant C^3\sqrt{T-t}$$

for all $t \in [0, T)$, where $\ell(t) := l_+(t) - l_-(t)$.

By translation, we may assume that

$$l_\pm(T) := \lim_{t \nearrow T} l_\pm(T) = 0.$$

Since $\dot{l}_+(t) < 0 < \dot{l}_-(t)$ for $t \in [t_*, T]$, we have

$$|l_\pm(t)| < \ell(t) \leqslant C\sqrt{T-t} \quad \forall t \in [t_*, T).$$