Some examples of global solutions associated with large initial data for the incompressible Navier-Stokes system

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Outline of the talk

1. The Cauchy problem
   - Presentation of the equations
   - Fundamental properties
   - Weak solutions
   - Strong solutions
   - Towards the largest adapted space
   - The Koch and Tataru space
2. Situations when large data can generate a global solution
   - Previous results
   - Two examples

Presentation of the equations

- Viscous, incompressible, homogeneous fluid, in two or three space dimensions
- Velocity $u = (u^1, u^2, u^3)(t,x)$, pressure $p(t,x)$

$$(NS) \qquad \begin{aligned} \partial_t u + u \cdot \nabla u - \Delta u &= -\nabla p, \\ \operatorname{div} u &= 0, \end{aligned}$$

with $\operatorname{div} u = \sum_{j=1}^{3} \partial_j u^j$ and $u \cdot \nabla u = \sum_{j=1}^{3} u^j \partial_j u = \sum_{j=1}^{3} \partial_j (u^j u)$.

Remark : The pressure can be eliminated by projection onto divergence-free vector fields :

$$\mathbb{P} = \mathrm{Id} - \nabla \Delta^{-1} \mathrm{div}.$$

Cauchy data : $u_{|t=0} = u_0$.

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Fundamental properties

Conservation of the energy

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 = 0$$

due to the structure of the nonlinear term : $(u \cdot \nabla u | u)_{L^2} = 0$.

Scale invariance

If u is a solution of (NS) associated with the initial data u_0 on $[0, T[$, then for all $\lambda > 0$,

$$u_\lambda(t,x) \stackrel{\text{def}}{=} \lambda u(\lambda^2 t, \lambda x)$$

is a solution associated with $u_{\lambda,0}(x) \stackrel{\text{def}}{=} \lambda u_0(\lambda x)$ on $[0, \lambda^{-2} T[$.

To solve (NS) one should try to use both informations.

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Weak solutions

Theorem [Leray, 1934]

Let $u_0 \in L^2(\mathbb{R}^d)$ be a divergence free vector field. There is a solution u of (NS) satisfying for all $t \geq 0$

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(t')\|_{L^2}^2 \, dt' \leq \|u_0\|_{L^2}^2.$$

In two space dimensions that solution is unique (and satisfies the energy equality).

Remarks :

- Proof by compactness.
- Uniqueness guaranteed if $u \in L^2([0,T]; L^\infty(\mathbb{R}^d))$..
- Search for conditions on the initial data to guarantee uniqueness.

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Strong solutions

One does not use the structure of the equation, but rather its scale invariance, by a fixed point method.

Solving (NS) is equivalent to solving

$$u = \mathbb{S}(t)u_0 + \mathbb{B}(u,u)$$

where $\mathbb{S}(t)$ is the heat semi-group on $\mathbb{R}^d$ and $\mathbb{B}$ the bilinear form

$$\mathbb{B}(u,u)(t) = -\int_0^t \mathbb{S}(t-t')\mathbb{P}\,\mathrm{div}\,(u\otimes u)(t')\,dt'.$$

The problem consists in finding an adapted Banach space X, such that $\mathbb{B}$ is continuous from $X\times X$ to X.

Remark : By the scale invariance, the norm on X must satisfy

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An existence and uniqueness result

Theorem

Let X be an adapted space. If u_0 is such that $\|\mathbb{S}(t)u_0\|_X$ is small enough, then there is a unique solution to (NS) in X.

Proof : This is simply Picard's theorem : if X is a Banach space, if $L \in \mathcal{L}(X)$ and if $\mathbb{B} \in \mathcal{B}(X)$, with $\|L\|_{\mathcal{L}(X)} < 1$, then for all x_0 in X satisfying

$$\|x_0\|_X < \frac{(1 - \|L\|_{\mathcal{L}(X)})^2}{4\|\mathbb{B}\|_{\mathcal{B}(X)}}$$

the equation

$$x = x_0 + Lx + \mathbb{B}(x, x)$$

has a unique solution in the ball centered at 0 and of radius $\dfrac{1 - \|L\|_{\mathcal{L}(X)}}{2\|\mathbb{B}\|_{\mathcal{B}(X)}}$.

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Some examples of adapted spaces with $d = 3$

- The space of functions u_0 such that $t^{\frac{1}{4}}\mathbb{S}(t)u_0$ belongs to $L^\infty([0,T];\dot H^1)$; this is guaranteed if u_0 is in $\dot H^{\frac{1}{2}}$ (Fujita-Kato, 1964).

- The space of functions u_0 such that $t^{\frac{1}{4}}\mathbb{S}(t)u_0$ belongs to $L^\infty([0,T];L^6)$; this is guaranteed if u_0 is in L^3 (Kato 1983).

- The space of functions u_0 such that $t^{\frac{1}{2}(1-\frac{3}{p})}\mathbb{S}(t)u_0$ belongs to $L^\infty([0,T];L^p)$, for p a real number in $]3,\infty[$;

  this corresponds to the Besov space $\dot B^{-1+\frac{3}{p}}_{p,\infty}$ (Cannone-Meyer-Planchon, 1993).

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Remarks

Those are small data or small time theorems.

They hold for the more general equation

$$\partial_t u - \Delta u + Q(u,u) = 0$$

where $Q(v,w) \stackrel{\text{def}}{=} \sum_{1 \leq j,k \leq 3} Q_{j,k}(D)(v^j w^k)$ and $Q_{j,k}(D)$ are smooth homogeneous Fourier multipliers of order 1.

Besov spaces and oscillations

Proposition

Let $f \in \mathcal{S}(\mathbb{R}^3)$, and $\sigma \in]0, 3(1-1/p)[$. We define

$$f_\varepsilon(x) \stackrel{\text{def}}{=} f(x) e^{i\frac{x_3}{\varepsilon}}.$$

Then

$$\|f_\varepsilon\|_{\dot{B}^{-\sigma}_{p,\infty}} \leq C\varepsilon^\sigma.$$

Proof : If $t \leq \varepsilon^2$, one has $\|\mathbb{S}(t) f_\varepsilon\|_{L^p} \leq \|f\|_{L^p}$ hence

$$\Big\| \|t^{\frac{\sigma}{2}} \mathbb{S}(t) f_\varepsilon\|_{L^p} \Big\|_{L^1([0,\varepsilon^2], \frac{dt}{t})} \lesssim \varepsilon^\sigma \|f\|_{L^p}.$$

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If $t \geq \varepsilon^2$, since $-i\varepsilon\partial_3 e^{i\frac{x_3}{\varepsilon}} = e^{i\frac{x_3}{\varepsilon}}$, one has

$$\begin{aligned}
\mathbb{S}(t)f_\varepsilon(x) &= -\varepsilon^2 \int_{\mathbb{R}^3} \partial_{y_3}^2 \left(G_t(x-y)f(y)\right) e^{i\frac{y_3}{\varepsilon}}\,dy \\
&= -\frac{\varepsilon^2}{t} \int_{\mathbb{R}^3} G_t(x-y)f(y)e^{i\frac{y_3}{\varepsilon}}\,dy \\
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where $G_t(x)$ is a function of the type $\dfrac{1}{(4\pi t)^{\frac{3}{2}}} e^{-\frac{|x|^2}{4t}} P(\dfrac{x}{\sqrt{t}})$, and the result follows from Young's inequality and integration in time.

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Towards the largest adapted space

Proposition [Meyer]

Let B be a Banach space continuously embedded in $\mathcal{S}'(\mathbb{R}^3)$ and such that

$$\lambda \|f(\lambda(\cdot - a))\|_B \sim \|f\|_B.$$

Then B is continuously embedded in $\dot{B}^{-1}_{\infty,\infty}$.

Proof : We notice that

$$|\langle f, e^{-|\cdot|^2} \rangle| \leq C\|f\|_B$$

By dilation and translation, we deduce that

$$\|f\|_{\dot{B}^{-1}_{\infty,\infty}} = \sup_{t>0} t^{\frac{1}{2}} \|\mathbb{S}(t)f\|_{L^\infty} \leq C\|f\|_B.$$

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The Koch and Tataru space

Definition

We denote by X the space of functions f of $L^2_{loc}(\mathbb{R}^+ \times \mathbb{R}^3)$ such that

$$\|f\|_X \stackrel{\text{def}}{=} \sup_{t>0}\Big(t^{\frac{1}{2}}\|f(t)\|_{L^\infty} + \sup_{\substack{x\in\mathbb{R}^3\\ R>0}} R^{-\frac{3}{2}}\Big(\int_{P(x,R)} |f(t,y)|^2 dydt\Big)^{\frac{1}{2}}\Big) < \infty,$$

where $P(x,R) = [0,R^2] \times B(x,R)$.

Remark : The space of initial data u_0 such that $\mathbb{S}(t)u_0$ is in X is the space BMO^{-1} and we have $\dot{B}^{-1}_{\infty,2} \hookrightarrow BMO^{-1} \hookrightarrow \dot{B}^{-1}_{\infty,\infty}$.

Theorem [Koch, Tataru 2001]

The operator $\mathbb{B}$ is continuous from $X \times X$ to X.

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Situations when large data can generate a global solution

- Spherical or helicoidal geometry (Ukhovskii, Yudovich 1968, Ladyzhenskaia 1969, Leibovich, Mahalov, Titi 1990, Ponce, Racke, Sideris, Titi 1994).
- Coriolis force (Babin, Mahalov, Nicolaenko 1996, Gallagher 1998, Chemin, Desjardins, Gallagher, Grenier 2001, Charve 2004).
- Thin domains (Raugel, Sell 1994, Iftimie, Raugel 2001).
- Perturbations of $2D$ vector fields (in the periodic case ; Iftimie 1997, Gallagher 1997).
- Perturbations of global solutions (Gallagher, Iftimie, Planchon 2003).
- Fourier transform of the data supported in “sum-closed frequency sets” far enough from zero (Giga, Inui, Mahalov, Saal, 2007).

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Two examples

- Example 1 : oscillations in one direction, and dilation in another (Chemin, Gallagher 2006)

  Link with the Montgomery-Smith toy Navier-Stokes model (Gallagher,Paicu 2008)

- Example 2 : slow variation in one direction (Chemin, Gallagher 2007)

Remark : Those results are specific to Navier-Stokes : the first one uses the structure of the nonlinearity, the second one uses the fact that the 2D equation is globally wellposed.

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Remark : Those results are specific to Navier-Stokes : the first one uses the structure of the nonlinearity, the second one uses the fact that the 2D equation is globally wellposed.

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A new, nonlinear smallness assumption

Theorem [Chemin, Gallagher 2006]

If u_0 is a smooth, divergence free vector field in $\mathbb{R}^3$ such that, for some (appropriate) space F

$$\|\mathbb{B}(\mathbb{S}(t)u_0, \mathbb{S}(t)u_0)\|_F \leq C \exp\Big(-C\|u_0\|^4_{\dot{B}^{-1}_{\infty,2}}\Big),$$

then the system (NS) has a unique global solution.

- F is a space for forcing terms in the Koch-Tataru theorem.
- **Proof :** define $R(t) = u(t) - \mathbb{S}(t)u_0$ then

$$R = \mathbb{B}(\mathbb{S}(t)u_0, \mathbb{S}(t)u_0) + 2\mathbb{B}(\mathbb{S}(t)u_0, R) + \mathbb{B}(R, R).$$

  and apply Picard's scheme again.
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Example 1

Theorem [Chemin, Gallagher 2006]

Let $\phi \in \mathcal{S}(\mathbb{R}^3)$ and $\alpha \in]0,1[$. Set

$$\varphi_\varepsilon(x) = \frac{(-\log\varepsilon)^{\frac{1}{5}}}{\varepsilon^{1-\alpha}} \cos\left(\frac{x_3}{\varepsilon}\right) \phi\Big(x_1, \frac{x_2}{\varepsilon^\alpha}, x_3\Big).$$

There is $C > 0$ such that, if ε is small enough,

$$u_0^\varepsilon(x) \stackrel{\text{def}}{=} (\partial_2\varphi_\varepsilon(x), -\partial_1\varphi_\varepsilon(x), 0)$$

satisfies

$$C^{-1}(-\log\varepsilon)^{\frac{1}{5}} \leq \|u_0^\varepsilon\|_{\dot{B}^{-1}_{\infty,\infty}} \leq C(-\log\varepsilon)^{\frac{1}{5}}, \quad \textit{and}$$

$$\|\mathbb{B}(\mathbb{S}(t)u_0^\varepsilon, \mathbb{S}(t)u_0^\varepsilon)\|_F \leq C\varepsilon^{\frac{\alpha}{3}}(-\log\varepsilon)^{\frac{2}{5}}.$$

In particular for ε small enough, u_0^ε generates a unique, global, smooth solution to (NS).

The Montgomery-Smith toy model

The equation

$$(TNS_1) \quad \partial_t u - \partial_x^2 u + |\partial_x|(u^2) = 0$$

has the same small data theorems as Navier-Stokes, but some large data can generate finite-time blow up (Montgomery-Smith, 2001).

Theorem [Gallagher, Paicu 2008]

Let $d = 2$ or 3. There is a bilinear operator Q, which is a d-dimensional matrix of Fourier multipliers, such as the equation

$$(TNS_d) \begin{cases} \partial_t u - \Delta u + Q(u,u) = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^d \\ \operatorname{div} u = 0 \\ u_{|t=0} = u_0 \end{cases}$$

preserves the **divergence free condition** and the **scale invariance** of (NS), and such that there is a global, unique solution if the data is small enough in BMO^{-1}. Moreover the initial data of the previous theorem generates a finite-time blow up.

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Example 2 : slow variation in one direction

Theorem [Chemin, Gallagher 2007]

Let $v_0^h = (v_0^1, v_0^2)$ and w_0 be smooth divergence free vector fields on $\mathbb{R}^3$. If ε is small enough, the initial data

$$u_0^\varepsilon(x) = \left(v_0^h + \varepsilon w_0^h, w_0^3\right)(x_h, \varepsilon x_3)$$

generates a unique, global solution u^ε of (NS).

Remark : One needs to check that such an initial data exists (that is not small) : let (f, g) be in $\mathcal{S}(\mathbb{R}^2)$ and $\mathcal{S}(\mathbb{R})$ respectively. Let us consider the function h^ε defined by $h^\varepsilon(x_h, x_3) \stackrel{\text{def}}{=} f(x_h)g(\varepsilon x_3)$. We have, if ε is small enough,

$$\|h^\varepsilon\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)} \geq \frac{1}{4}\|f\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^2)}\|g\|_{L^\infty(\mathbb{R})}.$$

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$$\|h^\varepsilon\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)} \geq \frac{1}{4}\|f\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^2)}\|g\|_{L^\infty(\mathbb{R})}.$$

Idea of the proof

The solution u^ε is written $u^\varepsilon = v^\varepsilon_{app} + R^\varepsilon$, where

$$v^\varepsilon_{app}(t,x) \stackrel{\mathrm{def}}{=} \left(\underline{v}^h + \varepsilon \underline{w}^{\varepsilon,h}, \underline{w}^{\varepsilon,3}\right)(t, x_h, \varepsilon x_3)$$

and $\underline{v}^h(t, x_h, y_3)$ solves the 2D Navier-Stokes system with data $v_0^h(x_h, y_3)$ (y_3 is a parameter) while $\underline{w}^\varepsilon$ solves a transport-diffusion equation

$$\partial_t \underline{w}^\varepsilon + \underline{v}^h \cdot \nabla_h \underline{w}^\varepsilon - \Delta_h \underline{w}^\varepsilon - \varepsilon^2 \partial_3^2 \underline{w}^\varepsilon = -(\nabla^h \underline{p}_1, \varepsilon^2 \partial_3 \underline{p}_1)$$

with data $w_0(x_h, y_3)$.

Claim : R^ε is small, globally in time.

$$\partial_t R^\varepsilon + R^\varepsilon \cdot \nabla R^\varepsilon + v^\varepsilon_{app} \cdot \nabla R^\varepsilon + R^\varepsilon \cdot \nabla v^\varepsilon_{app} - \Delta R^\varepsilon = -\nabla q_\varepsilon + F^\varepsilon,$$

where $F^\varepsilon(t, x_h, x_3) \sim \varepsilon^{\frac{1}{3}}$ in $L^2(\mathbb{R}^+; \dot H^{-\frac{1}{2}}(\mathbb{R}^3))$.

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Idea of the proof

Indeed $F^\varepsilon(t,x_h,x_3)$ is made of the following terms :

$$\varepsilon\Big((\varepsilon\underline{w}^\varepsilon\cdot\nabla\underline{w}^{\varepsilon,h},\underline{w}^\varepsilon\cdot\nabla\underline{w}^{\varepsilon,3})+(\underline{w}^\varepsilon\cdot\nabla\underline{v}^h,0)\Big)(t,x_h,\varepsilon x_3)$$

$$+\varepsilon\Big(\varepsilon(\partial_3^2\underline{v}^h,0)+(0,\partial_3\underline{p}_0)\Big)(t,x_h,\varepsilon x_3).$$

Study of the nonlinear terms : For any smooth a and b and any $1\leq j\leq 3$,

$$\|a\partial_j b\|_{L^2(\mathbb{R}^+;\dot H^{-\frac12}(\mathbb{R}^3))}\lesssim\|a\partial_j b\|_{L^2(\mathbb{R}^+;L^{\frac32}(\mathbb{R}^3))}\lesssim\|a\|_{L^\infty(\mathbb{R}^+;L^3(\mathbb{R}^3))}\|\partial_j b\|_{L^2(\mathbb{R}^+;L^3(\mathbb{R}^3))}.$$

This implies that

$$\|a\partial_j b(t,x_h,\varepsilon x_3)\|_{L^2(\mathbb{R}^+;\dot H^{-\frac12}(\mathbb{R}^3))}\lesssim\varepsilon^{-\frac23}\|a\|_{L^\infty(\mathbb{R}^+;\dot H^{\frac12}(\mathbb{R}^3))}\|\partial_j b\|_{L^2(\mathbb{R}^+;\dot H^{\frac12}(\mathbb{R}^3))}.$$

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The ill prepared case

Ideally one would like to consider initial data of the following type (with $0 < \alpha \leq 1$) :

$$u_0^\varepsilon(x) = \left(v_0^h + \varepsilon^\alpha w_0^h, \varepsilon^{\alpha-1} w_0^3\right)(x_h, \varepsilon x_3)$$

where $v_0^h = (v_0^1, v_0^2, 0)$ is a horizontal, smooth divergence free vector field on $\mathbb{R}^3$ and w_0 is a smooth divergence free vector field on $\mathbb{R}^3$.

Work in progress (Chemin, Gallagher, Paicu 2008) : consider

$$u_0^\varepsilon(x) = \left(\varepsilon^{\frac{1}{2}} w_0^h, \varepsilon^{-\frac{1}{2}} w_0^3\right)(x_h, \varepsilon x_3)$$

in $\mathbb{T}^2 \times \mathbb{R}$ with $\displaystyle\int_{\mathbb{T}^2} w_0^h(x_h, x_3) \equiv 0$, with (w_0^h, w_0^3) small and smooth enough (analytic-type in x_3) then there is a global solution.

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