Random modulation and persistence of solitons for the stochastic KdV equation

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the deterministic KdV equation

$$\partial_t u + \partial_x^3 u + \partial_x(u^2) = 0$$

- **"Universal model" :**
  Asymptotic model for long waves at the surface of water (small amplitude, shallow water, unidirectional propagation)
  Rigorous derivation : W. Craig, CPDE, 1985
  Model for plasma physics : Herman, J. Phys. A, 1990
- **Integrable equation**
  Hamiltonian system (action-angle variables) : infinite number of integrals of motion
  Allows to solve globally the equation (weakly) in spaces of irregular data : Kappeler-Toppalov, Duke Math. J., 2006
  $\rightsquigarrow$ white noise invariant measure

- **Inverse scattering methods :** Gardner, Green, Kruskal, Miura,PRL, 1967
  Resolution into solitons of any smooth enough and decaying solution : Ekhaus, Schuur, M2AS, 1983
  Solitons : two-parameter family of solutions $u_{c_0,x_0}(t,x) = \varphi_{c_0}(x - c_0 t + x_0)$ with

$$\varphi_{c_0}(x) = \frac{3c_0}{2\cosh^2(\sqrt{c_0}x/2)}$$

- **From PDE point of view :**
  - Equation globally well-posed in $H^s(\mathbb{R})$, $s > -3/4$, and $H^s(\mathbb{T})$, $s > -1/2$: Bourgain ; Kenig,Ponce,Vega ; Colliander, Staffilani, Takaoka, Tao
  - Solitons are orbitally stable Benjamin, Proc. Roy. Soc. Lond.,1972 and even asymptotically stable Pego, Weinstein, CMP, 1994 ; Martel, Merle, Arch. Rat. Mech. Anal., 01

Random perturbations of the KdV equation

No rigorous (mathematical) derivation

- **Forcing term :** Surface waves, random pressure e.g. turbulent wind on the surface
  $\rightsquigarrow$ add a term $\dot{\xi}(t,x)$ white noise in time
- **Variations of the bottom topography** modeled by a stationary (in x) random process
  $\rightsquigarrow$ add a term $(\partial_x u)\dot{\xi}(t)$ (white in time)
  Mathematically : KdV equation rewritten in the frame moving with velocity $\dot{\xi}$: $u(x-\xi(t))$
- **Random potential :** $u\dot{\xi}$ or $(\partial_x u)\dot{\xi}$
  Plasma physics, Herman, J. Phys. A, 1990

Mathematical results in the randomly perturbed case

- **Additive case :**

  PDE methods :

  dB, Debussche, Tsutsumi, JFA, 99 ; Printems, JDE, 99 :
  Equation is well posed in $H^s(\mathbb{R})$, $s > -5/8$ for irregular noise in space (includes "localised" space-time white noise)
  dB, Debussche, Tsutsumi, SIAM, 04 : Same result for $x \in \mathbb{T}$, but with $s > -1/2$, i.e. close to space-time white noise)
  dB, Debussche, Ann IHP, 07 : Random modulation of solitons

  Integrability methods (action-angle variables)
  Kuksin, Piatnitski, to appear in JMPA :
  Hasminski-Whitham averaging for

  $$\partial_t u + \partial_x^3 u - \nu \partial_x^2 u + \partial_x(u^2) = \sqrt{\nu}\dot{\xi}$$

  $\dot{\xi}$ white noise in time, regular in x ; modeling of weak turbulence

- **Multiplicative case :**

PDE Methods :

dB, Debussche, Int. Disc. Math. Sc., 07 : Existence of solutions in $L^2(\mathbb{R})$ and $H^1(\mathbb{R})$ (for noise in L^2 or H^1)

Tsutsumi, preprint 07 : Regular noise in space, $u(t,x)$ goes to zero a.s. as t goes to infinity, for any u_0, under specific assumptions on the noise, and $x \in \mathbb{T}$

Inverse scattering methods :

Garnier, J. Stat. Phys. 01 : Noise of the form $(\partial_x u)\dot{\xi}(t)$, $(\partial_x^3 u)\dot{\xi}(t)$ or $\partial_x(u^2)\dot{\xi}(t)$, i.e. perturbations of velocity, dispersion or nonlinearity ; propagation of solitons : equations on the scattering data ; no estimate on the remaining terms for the original solution

Original motivation of the work

Wadati, J. Phys. Soc. Japan, 1983 : Particular case of a noise that depends only on time

$$du + (\partial_x^3 u + \partial_x(u^2))dt = dW$$

with $W(t)$ real valued, centered, Brownian motion ; then $u(t,x) = U(t, x - \int_0^t W(s)ds) + W(t)$ with U solution of deterministic KdV. If in particular $U(t,x) = \varphi_{c_0}(x - c_0 t)$, then

$$\mathbb{E}(u(t,x)) = \mathbb{E}(\varphi_{c_0}(x - c_0 t - \int_0^t W(s)ds)) = \int_{\mathbb{R}} \varphi_{c_0}(x - c_0 t - y)\mu(dy)$$

with $\mu = \mathcal{L}(\int_0^t W(s)ds) \sim \mathcal{N}(0, t^3/3)$; one easily gets

$$\max_{x \in \mathbb{R}} \mathbb{E}(u(t,x)) \leq Ct^{-3/2}$$

Same result, but with $Ct^{-1/2}$ if noise $(\partial_x u) \circ dW/dt$

Is it possible to get such results for more general noises (depending on space) with, e.g., amplitude goes to zero ?

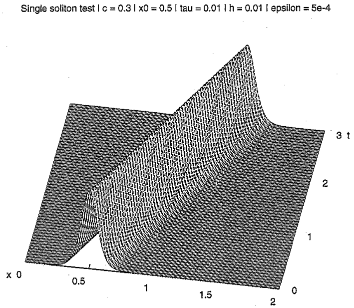


FIG. 3.2: *Propagation of a single soliton whose parameters are* $c = 0.3$, $x_0 = 0.5$ *for* $\varepsilon = 5.10^{-4}$ *on the time interval [0,3].*

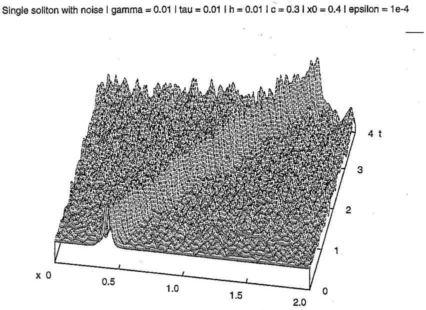
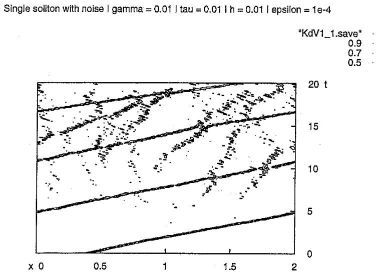
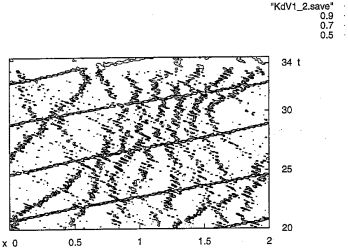


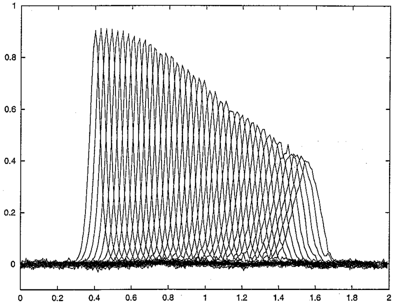
FIG. 3.13: *Single soliton with noise,* $\gamma = 10^{-2}$, $c = 0.3$, $\varepsilon = 10^{-4}$, $x_0 = 0.4$ *on* $[0, 3]$.


Single soliton with noise I gamma = 0.01 I tau = 0.01 I h = 0.01 I epsilon = 1e-4
"KdV1_1.save"
0.9
0.7
0.5
20 t
15
10
5
0
x 0
0.5
1
1.5
2

Single soliton with noise I gamma = 0.01 I tau = 0.01 I h = 0.01 I epsilon = 1e-4



$a)$



Description of the noise

$$du + (\partial_x^3 u + \partial_x(u^2))dt = \begin{cases} \varepsilon dW \\ \varepsilon u dW \\ \varepsilon(\partial_x u) dW \end{cases}$$

$W(t)$ infinite dimensional Wiener process i.e.

$$W(t,x) = \sum_j \Phi(e_j)(x) W_j(t)$$

$(W_j)_{j\in\mathbb{N}}$ independent family of real valued Brownian motions

$(e_j)_{j\in\mathbb{N}}$ complete orthonormal system of $L^2(\mathbb{R})$

- Φ Hilbert-Schmidt operator from $L^2(\mathbb{R})$ into $H^1(\mathbb{R})$, if additive noise
- $\Phi(e_j) = k * e_j$, with $k \in H^1(\mathbb{R}) \cap L^1(\mathbb{R})$ if multiplicative noise (W homogeneous or stationary in x)

In these cases there is a unique solution with paths a.s. in the energy space $H^1(\mathbb{R})$ **except for** $(\partial_x u)dW$

Exit from a neighborhood of the soliton : heuristics

Let $u_0(x) = \varphi_{c_0}(x)$ and let $\varepsilon > 0$ small ;

Question : Up to what time does the solution u^ε stay close to the soliton solution $\varphi_{c_0}(x - c_0 t)$?

Linearization of the deterministic equation :
If $u(t,x) = \varphi_{c_0}(x - c_0 t) + v(x - c_0 t)$, the linearized deterministic equation is

$$\partial_t v = \partial_x L_{c_0} v, \quad L_{c_0} = -\partial_x^2 + c_0 - 2\varphi_{c_0}\partial_x \varphi_{c_0}$$

Pego, Weinstein, Phil. Trans. Roy. Soc. Lond., 1992 : Spectrum of $\partial_x L_{c_0}$

- No unstable eigenvalue
- Generalized null space : $\begin{cases} \partial_x L_{c_0}(\partial_x \varphi_{c_0}) = 0 \\ \partial_x L_{c_0}(\partial_c \varphi_{c_0}) = -\partial_x \varphi_{c_0} \end{cases}$

  i.e. 0 is a simple degenerate eigenvalue with Jordan block

Heuristics :

Consider the linearization around $\varphi_{c_0}(x - c_0 t)$ of the stochastic equation ; projected on the "center manifold", i.e. on the generalized null space, the dynamics will formally be given by the SDE system :

$$\begin{cases} dX_1 = \varepsilon dW_1 \\ dX_2 = -X_1 dt + \varepsilon dW_2 \end{cases}$$

with (W_1, W_2) a $\mathbb{R}^2$-valued brownian motion (projection of $\varphi_{c_0} W$ on the null space) ; hence

$$X_2(t) = \varepsilon W_2(t) - \varepsilon \int_0^t W_1(s) ds;$$

but $\int_0^t W_2(s) ds \sim \mathcal{N}(0, t^3/3)$ and thus for large t,

$$\mathbb{P}(X_2(T) > \delta) \leq C \exp(-\frac{\delta}{\varepsilon^2 T^3})$$

This suggests that the time we are looking for ($X_2(T)$ small) is of the order of $\varepsilon^{-2/3}$

Rigorous results in the additive case

dB, Gautier, 07 : Consider the additive equation

$$\partial_t u + (\partial_x^3 u + \partial_x(u^2))dt = \varepsilon \Phi_n dW$$

with Φ_n a Hilbert-Schmidt operator from L^2 into H^1, $\|\Phi_n\|_{\mathcal{L}(L^2,H^1)} \leq 1$ for all n and $\lim_{n\to+\infty} \Phi_n(v) = (Id - \Delta)^{-1/2} v$ for all $v \in L^2$; define

$$\tilde{\tau}_\alpha^{n,\varepsilon} = \inf\{t \geq 0,\ \|u^{n,\varepsilon}(t,.+c_0 t) - \varphi_{c_0}\|_{H^1} > \alpha\}$$

Theorem : Let $c_0 > 0$, $\alpha > 0$ sufficiently small, then there exists a constant $C(\alpha, c_0)$ such that for all $T > 0$,

$$\liminf_{n\to+\infty} \liminf_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}(\tilde{\tau}_\alpha^{n,\varepsilon} \leq T) \geq -\frac{C(\alpha, c_0)}{T^3}$$

Hence, the soliton approximation is valid up to times at most of the order of $\varepsilon^{-2/3}$

Stability of the solitons :

The family $\{\varphi_{c_0}(.+x_0),\ x_0\in\mathbb{R}\}$ is a stable family of solutions of the deterministic equation

Benjamin, Proc. Roy. Soc. Lond. 1972 : Consider the functional $Q_{c_0}(u)=H(u)+c_0m(u)$ as a Lyapunov functional with

$$m(u)=\frac{1}{2}\int_{\mathbb{R}}u^2dx,\quad \text{and}\quad H(u)=\frac{1}{2}\int_{\mathbb{R}}u^2dx-\frac{1}{3}\int_{\mathbb{R}}u^3dx$$

Then $Q''_{c_0}=L_{c_0}$ satisfies $(L_{c_0}\eta,\eta)\geq|\eta|^2_{H^1}$, for all $\eta\in H^1(\mathbb{R})$ with $(\eta,\varphi_{c_0})=(\eta,\partial_x\varphi_{c_0})=0$.

$\rightsquigarrow$ the right scale in time should be ε^{-2}

Dynamics of the stochastic equation

Let $u^\varepsilon(0,x) = \varphi_{c_0}(x)$; in order to use the stability property of the soliton for (KdV), write the solution u^ε of the stochastic equation as

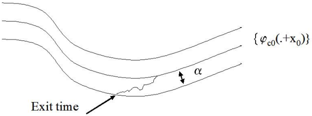
$$u^\varepsilon(t,x) = \varphi_{c^\varepsilon(t)}(x - x^\varepsilon(t)) + \varepsilon\eta^\varepsilon(t, x - x^\varepsilon(t))$$

where the parameters $x^\varepsilon(t)$ and $c^\varepsilon(t)$ are random modulation parameters, chosen such that for all t,
$(\eta^\varepsilon(t), \varphi_{c_0}) = (\eta^\varepsilon(t), \partial_x\varphi_{c_0}) = 0$.

This decomposition holds as long as $\|\varepsilon\eta^\varepsilon(t)\|_{H^1} \leq \alpha$ and $|c^\varepsilon(t) - c_0| \leq \alpha$ for $\alpha > 0$ sufficiently small.

Question : Can we estimate the time τ_α^ε with

$$\tau_\alpha^\varepsilon = \inf\{t > 0,\ \|\varepsilon\eta^\varepsilon(t)\|_{H^1} \geq \alpha \text{ or } |c^\varepsilon(t) - c_0| \geq \alpha\}?$$



Theorem dB, Debussche, 2007 : The stopping time (exit time) τ_α^ε satisfies : for all $\alpha > 0$ sufficiently small, there exists a constant C_α such that for all $T > 0$ and $\varepsilon > 0$ with $\varepsilon^2 T$ sufficiently small,

$$\mathbb{P}(\tau_\alpha^\varepsilon \leq T) \leq C_\alpha \varepsilon^2 T$$

Moreover, the modulation parameters $x^\varepsilon(t)$ and $c^\varepsilon(t)$, defined a.s. for $t \leq \tau_\alpha^\varepsilon$ are semi-martingales (solutions of a system of SDEs involving η^ε)

Remark : Easy to improve this estimate as :
$\mathbb{P}(\tau_\alpha^\varepsilon \leq T) \leq C_\alpha^k (\varepsilon^2 T)^k$, for any integer k. Actually, we have :

Theorem dB, Gautier, 2007 : Under the preceding assumptions, the exit time τ_α^ε satisfies : $\exists\ C(\alpha) > 0$, such that for all $T > 0$, and $\varepsilon > 0$ with $\varepsilon^2 T$ sufficiently small,

$$\mathbb{P}(\tau_\alpha^\varepsilon \leq T) \leq \exp(-\frac{C(\alpha)}{\varepsilon^2 T})$$

Further results in the additive case

Consider again the additive equation

$$\partial_t u + (\partial_x^3 u + \partial_x(u^2))dt = \varepsilon \Phi_n dW,$$

with Φ_n Hilbert-Schmidt from L^2 into H^1, approximating the operator $(Id - \Delta)^{-1/2}$; let $\tau_\alpha^{n,\varepsilon}$ be the exit time from the neighborhood of the randomly modulated soliton ;

Theorem dB, Gautier, 07 : Let $c_0 > 0$, $\alpha > 0$ sufficiently small, then there exists a constant $C(\alpha, c_0)$ such that for all $T > 0$,

$$\liminf_{n\to+\infty} \liminf_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}(\tau_\alpha^{n,\varepsilon} \leq T) \geq -\frac{C(\alpha, c_0)}{T}$$

Remark : Large deviation principle certainly also holds in the multiplicative case ; however, in order to obtain such estimate, one should solve a controlability problem by a potential $\rightsquigarrow$ open problem

Theorem : dB, Debussche, 2007

For any $T > 0$, there is a constant $C(T, \alpha, c_0)$ with

$$\mathbb{E}\left(\sup_{t \leq T \wedge \tau_\alpha^\varepsilon} \|\eta^\varepsilon(t)\|_{H^1}^2\right) \leq C(T, \alpha, c_0)$$

Moreover, η^ε converges to η as ε goes to zero, in $L^2(\Omega, L^\infty(0, T \wedge \tau_\alpha^\varepsilon, L^2(\mathbb{R})))$ and η is the solution of the linear equation

$$d\eta = \partial_x L_{c_0} \eta dt + Q(\varphi_{c_0} d\tilde{W})$$

with $\eta(0) = 0$ and $\tilde{W}(t, x) = W(t, x + c_0 t)$ is the noise translated in the frame moving with velocity c_0 ; Q : projector on $\text{span}\{\varphi_{c_0}, \partial_x \varphi_{c_0}\}^\perp$

Remark : (multiplicative noise)

- $\mathcal{L}(\tilde{W}) = \mathcal{L}(W)$ due to the homogeneity of the noise
- η is a centered Gaussian process ("Ornstein-Uhlenbeck" if $\partial_x L_{c_0}$ dissipative operator)

Pego, Weinstein, CMP, 1994 : Asymptotic stability of solitary waves in spaces with exponential weights

$$H^1_a = \{v \in H^1(\mathbb{R}),\ e^{ax}v \in H^1(\mathbb{R})\}, \quad \|v\|_{H^1_a} = \|e^{ax}v\|_{H^1}$$

Then $\exists b > 0$, $\exists C > 0$ (depending on a, c_0) such that

$$\|e^{-t\partial_x L_{c_0}} Qw\|_{H^1_a} \le Ce^{-bt}\|Qw\|_{H^1_a}, \quad \forall w \in H^1_a$$

In H^1_a, the covariance operator of η satisfies

$$\begin{aligned} \mathrm{tr}(\Phi(t)) &= \sum_j \int_0^t \|e^{-(t-s)\partial_x L_{c_0}} Q[\varphi_{c_0}(k * e_j)]|^2_{H^1_a} ds \\ &\le C\left(\int_0^t e^{-2b\sigma} d\sigma\right) \sum_j \|Q[\varphi_{c_0}(k * e_j)]\|^2_{H^1_a} \\ &\le C|\varphi_{c_0}|^2_{H^1_a} |k|^2_{H^1} \end{aligned}$$

Hence η converges weakly to a Gaussian stationary measure as t goes to infinity

- The modulation equations are given by

$$\begin{cases} dx^\varepsilon = c_0 dt + \varepsilon B_1 dt + \varepsilon dB_2 + o(\varepsilon) \\ dc^\varepsilon = \varepsilon dB_1 + o(\varepsilon) \end{cases}$$

  with (B_1, B_2) a $\mathbb{R}^2$-valued brownian motion, corresponding to $P\varphi_{c_0}\tilde{W}(t,x)$, with $P = Id - Q$;
- Note that coupling with η^ε only at next order (true only in the multiplicative case)
- Keeping only order one in ε, the process $(c^\varepsilon(t) - c_0, x^\varepsilon(t) - c_0 t)$ is an $\mathbb{R}^2$-valued centered Gaussian process
- Keeping only first order terms in ε, one may use the same computations as Wadati, and using $\varphi_c = c\varphi_1(\sqrt{c}x)$, and $\varphi_1 \in L^1(\mathbb{R})$, we obtain

$$\max_{x\in\mathbb{R}} \mathbb{E}\left(\varphi_{c^\varepsilon(t)}(x - x^\varepsilon(t))\right) \leq K_{c_0}\varepsilon^{-1/2}t^{-5/4}$$

  for large t

Proof of the lower bound on exit time (additive case)

Recall that we want to prove :

There exists a constant $C(\alpha, c_0)$ such that for all $T > 0$,

$$\liminf_{n\to+\infty} \liminf_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}(\tilde{\tau}_\alpha^{n,\varepsilon} \leq T) \geq -\frac{C(\alpha, c_0)}{T^3}$$

The laws $\left(\mu^{u^{n,\epsilon,\varphi_{c_0}}}\right)_{\epsilon>0}$ satisfy a LDP of speed ε^2 and good rate function

$$I^n(w) = \frac{1}{2} \inf_{h\in L^2(0,T;L^2):\ w=\mathbf{S}^{n,\varphi_{c_0}}(h)} \|h\|^2_{L^2(0,T;L^2)}$$

where $\mathbf{S}^{n,\varphi_{c_0}}(h)$ is the unique mild solution in X_T of the control problem

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x(u^2) = \Phi_n h, \\ u(0) = \varphi_{c_0} \text{ and } h \in L^2\left(0, T; L^2\right). \end{cases}$$

It means that for any Borel set B of $C([0,T];H^1)$,

$$-\inf_{w\in\overset{\circ}{B}} I^n(w)\leq \underline{\lim}_{\varepsilon\to 0}\varepsilon^2\log\mathbb{P}\left(u^{n,\varepsilon,\varphi_{c_0}}\in B\right)$$

$$\overline{\lim}_{\varepsilon\to 0}\varepsilon^2\log\mathbb{P}\left(u^{n,\varepsilon,\varphi_{c_0}}\in B\right)\leq -\inf_{w\in\overline{B}} I^n(w).$$

We then need to construct controls h such that $S^{n,\varphi_{c_0}}(h)\in B$ (the set of functions leaving the α-neighborhood of the soliton before T), and such that the infimum is bounded below by $-\frac{C(\alpha,c_0)}{T^3}$

First replace ϕ_n by $(Id-\Delta)^{-1/2}$ and denote $S(h)$ the corresponding control map

We consider controls h such that $S(h)=\varphi_{c(t)}(.-\int_0^t c(s)ds)$, so that $h=-c'(t)\partial_c\varphi_{c(t)}(.-\int_0^t c(s)ds)$, $c(0)=0$ and the condition

$$\|S(h)(T)-\varphi_{c_0}(.-c_0T)\|_{H^1}>\frac{3}{2}\alpha$$

This latter condition is actually satisfied if

$$\frac{3}{2}c_0 - \varphi_{c(T)}(. - \int_0^T (c_0 - c(s))ds) > 3C_\infty\alpha$$

with $\|v\|_{L^\infty} \leq C_\infty \|v\|_{H^1}$ for all v ;

This is in turn satisfied if $\int_0^T (c_0 - c(s))ds > \delta(c_0, \alpha)$

The energy to minimize is

$$\frac{1}{2}\int_0^T \|c'(t)(I-\Delta)^{1/2}\left[\partial_c \varphi_{c(t)}(. - \int_0^t c(s)ds)\right]\|_{L^2}^2 dt$$
$$= \frac{1}{2}\int_0^T (c'(t))^2 g(c(t))dt$$

We then get the expected bound by taking $c(t) = c_0 - 3t\delta/T^2$

For ϕ_n, we use a continuity argument on the control map and get the same estimate as n goes to infinity.

random modulation : parametrization

- The proof uses a local parametrization $u \mapsto (c(u), x(u))$ of the soliton profile, such that $u = \varphi_{c(u)}(. - x(u)) + R(u)$ where R satisfies the orthogonality conditions $(R(u), \varphi_{c_0}) = (R(u), \partial_x \varphi_{c_0}) = 0$
- The parametrization is obtained thanks to the implicit function Theorem, and holds for u in a α-neighborhood of the orbit of φ_c, with $|c - c_0| \leq \alpha$
- It is sufficient to estimate $\mathbb{P}(\bar{\tau}_\alpha^\varepsilon < T)$ with

$$\bar{\tau}_\alpha^\varepsilon = \inf\{t \geq 0, |c^\varepsilon(t) - c_0| \geq \alpha, \ \|u^\varepsilon(t, . + x^\varepsilon(t)) - \varphi_{c_0}\|_{H^1} \geq \alpha\}$$

  where we have set $x^\varepsilon(t) = x(u^\varepsilon(t))$ and $c^\varepsilon(t) = c(u^\varepsilon(t))$
- We make use of the Lyapunov functional $Q_{c_0}(u^\varepsilon) = H(u^\varepsilon) + c_0 m(u^\varepsilon)$, thanks to the equations for $H(u^\varepsilon(t))$ and $m(u^\varepsilon(t))$, in order to estimate $\|u^\varepsilon(t, . + x^\varepsilon(t)) - \varphi_{c_0}\|_{H^1}$

random modulation : upper bounds for the exit time

Manipulations of the Lyapunov functional Q_{c_0} give

$$|c^\varepsilon(\tau_\alpha^\varepsilon) - c_0|^2 \leq C \left[\|\varepsilon\eta^\varepsilon(\tau_\alpha^\varepsilon)\|_{L^2}^4 + 4\varepsilon^2 \left| \int_0^{\tau_\alpha^\varepsilon} ((u^\varepsilon)^2, dW(s)) \right|^2 \right.$$

$$\left. +4\varepsilon^4 |k|_{L^2}^4 \left(\int_0^{\tau_\alpha^\varepsilon} |u^\varepsilon(s)|_{L^2}^2 ds \right)^2 \right]$$

$$\|\varepsilon\eta^\varepsilon(\tau_\alpha^\varepsilon)\|_{H^1}^2 \leq C \left[\|\varepsilon\eta^\varepsilon(\tau_\alpha^\varepsilon)\|_{L^2}^4 + \varepsilon \left| \int_0^{\tau_\alpha^\varepsilon} (\partial_x u^\varepsilon, \partial_x(u^\varepsilon \phi_{c_0}) dW(s)) \right| \right.$$

$$+ \varepsilon \left| \int_0^{\tau_\alpha^\varepsilon} ((u^\varepsilon)^3, dW) \right| + c_0 \varepsilon \left| \int_0^{\tau_\alpha^\varepsilon} ((u^\varepsilon)^2, dW(s)) \right|$$

$$+ 4\varepsilon^2 \left| \int_0^{\tau_\alpha^\varepsilon} ((u^\varepsilon)^2, dW(s)) \right|^2 + \varepsilon^2 |k|_{H^1}^2 \int_0^{\tau_\alpha^\varepsilon} |u^\varepsilon|_{H^1}^2 ds$$

$$\left. +\varepsilon^2 |k|_{L^2}^2 \int_0^{\tau_\alpha^\varepsilon} |u^\varepsilon|_{H^1}^3 ds + \varepsilon^4 |k|_{L^2}^4 \left(\int_0^{\tau_\alpha^\varepsilon} |u^\varepsilon|_{L^2}^2 ds \right)^2 \right]$$

- Use of the Cauchy-Schwarz + Burkhölder inequalities to estimate the stochastic integrals
  $\rightsquigarrow$ obtain $\mathbb{P}(\tau_\alpha^\varepsilon < T) \leq C_\alpha \varepsilon^2 T$

- Use of exponential tail estimates for those integrals
  $\rightsquigarrow$ get the bound $\mathbb{P}(\tau_\alpha^\varepsilon < T) \leq \exp(-\frac{C(\alpha)}{\varepsilon^2 T})$

Remark : The same kind of estimates (upper bounds) hold for a solution which starts from a sum of n solitary waves (or n-solitons) dB, El Dika, 2006)

concluding remarks

- We have considered small random perturbations, white in time, of the KdV equation with the soliton as initial data
- The time scale on which solution stays in a neighborhood of the soliton for an additive perturbation is at most $\varepsilon^{-2/3}$, confirming the heuristic.
- Like for asymptotic stability under perturbation of the initial datum, modulations help to understand the persistence of solitons.
- The time scale on which the solution stays in a neighborhood of the randomly modulated soliton is ε^{-2}.
- A central limit theorem holds, i.e. the order one part of the remaining term converges as ε goes to 0 to a centered Gaussian process which has an invariant measure in H^1_a
- Keeping only the order one in the modulation equations, the soliton diffuses at rate $t^{-5/4}$
- What about larger time scales than ε^{-2} ? ex : Tsutsumi, 07, Garnier, 01...