Uniqueness of Mean field equations: Known results, Open problems and Applications.

Chang-Shou Lin

Department of Mathematics
National Taiwan University

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1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^2$, and $H_0^1(\Omega)$ = the completion of $C_0^1(\Omega)$under the norm

$$\|\nabla u\| = (\int_\Omega |\nabla u|^2 dx)^{\frac{1}{2}}$$

The Moser-Trudinger inequality says that there is a constant $c = c(\Omega)$, such that

$$\int_\Omega \exp(4\pi \frac{(u)^2}{\|\nabla u\|^2})dx \leq C.$$

Since $u \le (\frac{\|\nabla u\|}{2\sqrt{4\pi}})^2 + (\frac{\sqrt{4\pi}u}{\|\nabla u\|})^2$,

$$\begin{aligned}
\int_\Omega e^u dx &\le \int_\Omega \exp(\frac{\|\nabla u\|^2}{16\pi} + 4\pi\frac{u^2}{\|\nabla u\|^2})dx \\
&\le \exp(\frac{\|\nabla u^2\|}{16\pi}) \cdot \int_\Omega \exp(4\pi\frac{u^2}{\|\nabla u\|^2})dx \\
&\le c\exp\frac{\|\nabla u\|^2}{16\pi}
\end{aligned}$$

i.e.

$$\frac{1}{2}\int_\Omega |\nabla u|^2 dx - 8\pi\int_\Omega e^u dx \ge c' \tag{1}$$

For $\rho > 0$, we consider

$$J_\rho(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \rho\int_\Omega e^u dx$$

Inequality (2) yields that $J_\rho(u)$ is bounded from below provided that $\rho \leq 8\pi$. If $\rho < 8\pi$, u_k be a minimizing sequence, i.e.

$$\begin{aligned} J_\rho(u_k) &= \frac{1}{2}\int |\nabla u_k|^2 - \rho \int_\Omega e^{u_k} dx \longrightarrow \inf_{H_0^1(\Omega)} J_\rho(u) \\ &= (\frac{1}{2} - \frac{\rho}{16\pi}) \int |\nabla u_k|^2 \\ &\quad +\rho(\frac{1}{16\pi}\int |\nabla u_k|^2 - \int_\Omega e^{u_k} dx) \\ &\geq (\frac{1}{2} - \frac{\rho}{16\pi}) \int |\nabla u_k|^2 dx + c \end{aligned}$$

i.e.

$$\int_\Omega |\nabla u_k|^2 dx \leq c_1$$

Then by the Moser-Trudinger inequality, $\exp c_0\, u_k^2 \in L^1(\Omega)$. Thus, a subsequence of $\{u_k\}$ can be chosen such that $u_k \rightharpoonup u$ in $H_0^1(\Omega)$ and

$$\int_\Omega e^{u_k(x)} dx \longrightarrow \int_\Omega e^{u(x)} dx. \text{ Then } \ u \in H_0^1(\Omega)$$

satisfies

$$J_\rho(u) = \inf_{H_0^1(\Omega)} J(u).$$

Obviously, the Euler-Lagrange equation for J_ρ is

$$\begin{cases} \Delta u + \frac{\rho e^u}{\int e^u} = 0 \text{ in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

In this lecture, I consider the equation:

$$\begin{cases} \Delta u + \rho \dfrac{h(x)e^u}{\int h(x)e^u} = \sum_{k=1}^{N} \alpha_k \delta_{P_k}, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{2}$$

where h(x) is a positive C^1 function, ρ is a constant positive. $\alpha_k > 0$ and δ_{P_k} is the Dirac measure at $P_k \in \Omega$.

We are also interested in the situation when the space is a compact Riemann surface:

$$\Delta u + \rho\Big(\frac{h(x)e^u}{\int_M h(x)e^u} - \frac{1}{|M|}\Big) = \sum \alpha_k\Big(\delta_{Pk} - \frac{1}{|M|}\Big), \text{ in } M, \tag{3}$$

where $|M|$ is the area of M.

Equation (2)or(3) is called mean field equation because it often arises in the context of statistical mechanics of point vortices in the mean field limit. See the works by Caglioti-Lions-Marchioro-Pulvirenti, Kiessling, Polani-Dritschel...etc.

Equation (2)or(3)have appeared in many different research field. In the conformal geometry, when M is S^2, and $p = 8\pi$, the equation is related to the Nirenberg problem. For a given positive function $h(x)$, we want to find a new metric $ds^2 = e^u \|dx\|^2$ such that $h(x)$ is the Gaussian causative of ds^2. In physics, equations (2) and (3) is also related to many physics models from Gauge field theory.

Equation (2) and (3) has been extensively studied for the past three decades, many results on existence has been shown.
For example, Chen and Lin have proved the following theorems.

Theorem 1 *suppose $\rho \neq 8\pi m$, m is a positive integer. Then the Leray-Schauder degree d_ρ for equation(1) is well-defined and*

$$d_\rho = \begin{cases} 1 & \text{if } \rho < 8\pi \\ \frac{g(g+1)\cdots(g+m-1)}{m!} & \text{if } 8\pi m < \rho < 8\pi(m+1) \end{cases}$$

where $g =$ the number of "holes" of Ω.

Theorem 2 *Suppose $\rho \neq 8m\pi$, m is a positive integer. Then the Leray-Schauder degree d_ρ for equation (3) is well-defined, and*

$$d_\rho = \begin{cases} 1 & \text{if } \rho < 8\pi \\ \frac{(1-\chi(M))(2-\chi(M))\cdots(m-\chi(M))}{m!} & \text{if } 8m\pi < \rho < 8(m+1)\pi \end{cases}$$

where $\chi(M)$ is the Euler characteritic of M.

3. Uniqueness for equation

We first consider equation in bounded domains of $\mathbb{R}^2$, without singularity data, i.e. $\alpha_k = 0$.

Theorem 3 *suppose* Ω *is a bounded* C^2 *domain in* $\mathbb{R}^2$, *and* $\rho \leq 8\pi$, *then equation (2) with* $\alpha_k = 0$ *possesses at most one solution.*

When Ω is a simply connected bounded domain in $\mathbb{R}^2$ and $\rho < 8\pi$, Theorem 1 was proved by Nagasaki and Suzuki in 1988. Under the same assumption of Ω and $\rho = 8\pi$, Theorem 1 was proved by Chen, Chang and Lin. For a general domain in $\mathbb{R}^2$, i.e. without the assumption of simple-connectedness, Theorem 1 was recently proved by C.S. Lin.

The paper by Nagasaki and Suzuki is the first one to show the uniqueness of mean field equation, They used the classic Bol isoparametric inequality and symmetrization method (due to C. Bandle) to prove the theorem. Their method becomes a standard tool in the proof of uniqueness problem.

Given a continuous function u on Ω and for any subdomain $\omega \Subset \Omega$, set

$$m(\omega) = \int_{\omega} e^{u} dx, \ \ell(\partial\omega) = \int_{\partial\omega} e^{\frac{u}{2}} ds$$

Lemma 1 *Let Ω be a simply-connected domain and $u \in C^2(\Omega)$ satisfy*

$$\Delta u + e^u \geq 0 \text{ in } \Omega.$$

If $\int_\Omega e^u dx \leq 8\pi$, then the inequality

$$2\ell^2(\partial\omega) \geq m(\omega)(8\pi - m(\omega))$$

holds for any subdomain $\omega \Subset \Omega$.

Remark:

1. Let $ds^2 = e^u|dx|^2$ be a conformal metric of flat metric in Ω. The inequality $\triangle u + e^u \geq 0$ in Ω is equivalent to saying the Gauss curvature of $ds^2 \leq 1$ on Ω.

2. The assumption Simple-connectedness of Ω is essential, and can not be removed.

Let $u \in C^2(\Omega)$ satisfy

$$\Delta u + e^u \geq 0 \ \ in \ \ \Omega, and \int_\Omega e^u dx \leq 8\pi$$

where Ω is simply-connected. For any $\omega \subset \Omega$, let

$\lambda_1 =$ *the first eigenvalue of* $\Delta + e^u$ *with Dirichlet boundary value on* $\partial\omega$

Lemma 2 *If $\lambda_1 \leq 0$, i.e $\exists\, \varphi \in H_0^1(\omega)$,*

$$\int_\omega |\nabla\varphi|^2 - \int_\omega e^u\varphi^2 \leq 0,$$

then $\int_\omega e^u \geq 4\pi$.

Lemma 2 is basically due to C. Bandle, and is a consequence of Lemma 1 and the symmetrization.

Remark: In Lemma 2, Ω is required to be simply- connected, but the subdomain ω is not necessarily simply- connected.

Proof of Theorem 1: By using Lemma 2, Nagasaki-Suzuk proves the linearized equation at u is non-degenerate, i.e. If φ satisfies

$$\begin{cases} \Delta\varphi + \rho\frac{e^u\varphi}{\int e^u} - \frac{(\int e^u\varphi)^2}{(\int e^u)^2} = 0, \ \text{ in } \ \Omega \\ \varphi = 0, \ on\ \partial\Omega, \end{cases}$$

then $\varphi \equiv 0$ in Ω.

For simplicility, we assume

$$\int_\Omega e^u \varphi = 0$$

Then φ satisfies

$$\Delta\varphi + e^v \varphi = 0.$$

$$\Omega^+ = \{x \mid \varphi(x) > 0\},\ \Omega^- = \{x \mid \varphi(x) < 0\}.$$

Let $v = u + \log\rho - \log\int_\Omega e^u$ then

$$\Delta v + e^v = 0$$

By Lemma 2,

$$\int_{\Omega^+} e^v = \rho\int_{\Omega^+} \frac{e^u}{\int_\Omega e^u} \geq 4\pi \text{ and}$$
$$\int_{\Omega^-} e^v = \rho\int_{\Omega^-} \frac{e^u}{\int_\Omega e^u} \geq 4\pi$$

i.e.

$$\rho = \rho \int_{\Omega^+} \frac{e^u}{\int_\Omega e^u} + \rho \int_{\Omega^-} \frac{e^u}{\int_\Omega e^u} \geq 8\pi,$$

yields a contradiction if $\rho < 8\pi$.

Uniqueness follows immediately from the non-degeneracy of the linearized equation.

Theorem 4 (*Batolluci and Lin*) *Suppose* Ω *is simply-connected domain and* $\alpha_j > 0$*. Then equation (2) possesses at most one solution of equation (2) provided that* $\rho \leq 8\pi$*.*

Question 1: Does the uniqueness hold for equation (2) if $\rho \leq 8\pi$.

Let $G(x,\ p)$ be the Green function of Δ and $\gamma(p)$ is the regular part of the Green function, i.e. $\forall p \in \Omega$

$$\widetilde{G}(x,\ p) = G(x,\ p) + \frac{1}{2\pi}\log|x-p|$$
$$\gamma(p) = \widetilde{G}(p,\ p)$$

Assume p_0 is a critical point of $\gamma(p)$ set

$$\begin{aligned} H(y) &= e^{8\pi(\widetilde{G}(y,p)-\gamma(p))} - 1 \ \text{ for } \ y \in \Omega \\ &= \sum a_{ij}(y_i - p_{0i})(y_j - p_{0j}) + O(|y-p_0|^3), \end{aligned}$$

where $a_{11} + a_{22} = 0$.

Set

$$D(p_0) = \lim_{\varepsilon \longrightarrow 0} \int_{\Omega \setminus B(p_0,\ \varepsilon)} \frac{H(y)}{|y-p_0|^4}dy - \int_{\Omega^c} \frac{dy}{|y-p_0|^4}$$

The following result is an application of Theorem 1.

Theorem 5 *Let Ω be a C^2 domain. Then equation (2) with $\rho = 8\pi$ possesses a solution if and only if $D(p_0) > 0$ for some maximum point p_0 of γ.*

When Ω is a simply connected domain $D(p_0)$ can be expressed by the conformal mapping:

$$f : \; B_1 \longrightarrow \Omega, \;\; f(0) = p_0$$

the assumption $\gamma'(p_0)$ implies $f''(0) = 0$. By Taylor expansion,

$$f(z) = f(0) + a_1 w + \sum_{n=3}^{\infty} a_n w^n$$

Then

$$D(p_0) = \sum_{n=3}^{\infty} \frac{n^2}{n-2} |a_n|^2 - |a_1|^2$$

Example:

$$\Omega = B(0;\ 1), \quad p_0 = 0$$

$$f(z) = z \ \ i.e. \ \ D(0) = -1 < 0$$

Equation (2) with $\rho = 8\pi$ has no solution when $\Omega = B_0(0;\ 1)$.

Corollary 1 *Suppose $\gamma(p)$ has more than one maximum points. Then equation (2) with $\rho = 8\pi$ possesses one solution.*

In fact, we can prove that if q is a critical point of γ, and $D(q) \leq 0$, then q must be a maximum point of γ, furthermore, q is the unique maximum point.

Next, we consider the case when $\alpha_j > 0$.

Question 1: What is the set of $(p_1, \cdots, p_N)$ when there exists a solution of equation (2) with $\rho = 8\pi$?

4. case of S^2

In the case of S^2, the Moser-Trudimger inequality can read as: For $\rho \leq 8\pi$

$$\begin{aligned} J_\rho(u) &= \frac{1}{2}\int_{S^2}|\nabla u|^2 - \rho \log \int_{S^2} e^u \geq -c, \ \forall u \in H^1(S^2), \\ H^1(S^2) &= \{u \in H^1(S^2) | \int_{S^2} u = 0\} \end{aligned}$$

Aubin improved the inequality, when $u \in H^1(S^2)$ satisfies

$$\int_{S^2} x_j e^u = 0 \ \ j = 1,\ 2,\ 3, \tag{4}$$

then for $\rho < 16\pi$

$$J_\rho = \frac{1}{2}\int_{S^2}|\nabla u|^2 - \rho \log \int_{S^2} e^u \geq -c_\rho.$$

The famous Onofri theorem says

$$J_\rho(u) \geq 0 \ \ \forall u \in H^1(S^2) \ \text{ and } \rho \leq 8\pi.$$

S.Y.A. Chang and P. Yang ask the following question:
Suppose $u \in H^1(S^2)$ and satisfies (4). Then

$$J_\rho(u) \geq 0 \ \text{ for } \ \rho \leq 16\pi.$$

To answer the question, we consider a minimizer of $J_\rho(u)$, Chang and Young found that any minimizer u of J_ρ under the conditions (4), satisfies

$$\Delta u + \rho\left(\frac{e^u}{\int_{S^2} e^u} - \frac{1}{4\pi}\right) = 0 \ \text{ in } \ S^2. \tag{5}$$

We want to prove the trivial solution u is the only solution of (5) provided that $\rho \leq 16\pi$ and $\rho \neq 8\pi$. Some partial results has been obtained.

Theorem 6 *If additionally, u is axially symmetric with respect to one direction, then $u \equiv 0$.*

This was proved by Feldman-Frose-Ghoussoub-Gui, Gui-Wei, Lin.

Theorem 7 *If $\rho \leq 12\pi$, then $u \equiv 0$.*

Fix any direction ξ . By using the stereographic projection $\pi : S^2 \longrightarrow \mathbb{R}^2$ with respect to ξ, we can reduce equation (5) to the following:

$$\Delta v(x) + (1 + |x|^2)^\ell e^{v(x)} = 0 \text{ in } \mathbb{R}^2, \tag{6}$$

where $v(x) = u(\pi^{-1}(x)) + \dfrac{\rho}{4\pi} \log(1 + |x|^2) + c$, and $\ell = \dfrac{\rho}{4\pi} - 2$.

If $\rho < 8\pi$, then $\ell < 0$. The method of moving planes implies $v(x) = v(|x|)$, which means u is axially symmetric with respect to any direction on S^2. Thus $u \equiv \text{const} \;\Rightarrow\; u \equiv 0$. (This is the Onofri theorem).

If $\rho > 8\pi$, then $\ell > 0$. The method of moving planes fails to show that $v(x)$ is radially symmetric. For $\rho \leq 12\pi$, we can choose some particular direction ξ, and by using the stereographic projection with respect to ξ, equation (4) reduces to equation (5). Set $\phi = y\dfrac{\partial v}{\partial x} - x\dfrac{\partial v}{\partial y}$, ϕ satisfies the linearized equation

$$\Delta\phi + (1 + |x|^2)^{\ell} e^{v}\phi = 0 \;\text{ in } \mathbb{R}^2.$$

We prove that

if $\phi \neq 0$, then the nodal line of ϕ divides that $\mathbb{R}^2$ at least into 3 components $\Omega_1,\ \Omega_2, \Omega_3$.

Since S^2 is a simply connected, and Lemma 2 can be applied to the potential $(1+|x|^2)^\ell e^u$. Thus

$$\int_{\Omega_j} (1+|x|^2)^\ell e^{v(x)} dx \geq 4\pi, \ \ j = 1,\ 2,\ 3.$$

i.e.

$$\rho = \int_{\mathbb{R}^2} (1+|x|^2)^\ell e^{v(x)} dx \geq 12\pi.$$

So, if $\rho \leq 12\pi$, then $\phi \equiv 0$, i.e $u(x)$ is axially symmetric with some direction. Then Theorem 5 follows from Theorem 4.
A related result is the following aprion estimate.

Theorem 8 *There is a constant C such that for any solution u of equation (5) with $8\pi \neq \rho \leq 16\pi$,*

$$|u(x)| \leq C.$$

5. the case of torus

We consider two mean field equations of mean field on a tours T.

$$\Delta u + \rho(\frac{e^u}{\int e^u} - \frac{1}{|T|}) = 0 \tag{7}$$

or

$$\Delta u + \rho \frac{e^u}{\int e^u} = \rho\delta_0 \tag{8}$$

where δ_0 is the Dirac measure at $O \in T$.
Let $0 < b \leq a$, $T = \{(x,\ y) \mid |x| \leq \frac{a}{2},\ |y| \leq \frac{b}{2}\}$. The linearized operator at $u = 0$ is $\Delta + \rho/T$. Obviously, if $\rho > \lambda_1(T)|T|$, then 0 is not a local minimizer. In this case, there exists a solution $u(x,\ y) = u(x)$, which bifurcated from $\rho = \lambda_1(T)|T|$ and the trivial solution. So, we want to know whether this one-dimensional solution is the only nontrivial solution when $\rho \leq 8\pi$.

Theorem 9 *Suppose $\rho \le 8\pi$. Then any minimizer of equation (7) must be independent of y variable. Furthermore, if $\rho \le \lambda_1(T)|T|$, then $u \equiv 0$.*

Remark:

1. By using the method of moving planes, it can be shown that any minimizer is symmetric with respect to some points. Theorem 7 is also proved by the Bol isoparametric inequality and Lemma 2. However, because the torus is not simply-connected, it is very subtle to use those techniques for the proof of Theorem 7.
2. the question whether any solution of (7) is one-dimensional remains open.
3. If the fundamental cell of T is not a rectangle, then it is an open problem whether the conclusion of Theorem 4 holds.

For equation (8), we consider $T = \{sw_1 + tw_2 \mid |s| \le \frac{1}{2},\ |t| \le \frac{1}{2}\}$, where $w_1,\ w_2$ are linearly independent in $\mathbb{R}^2$.

Theorem 10 *For $\rho \in [4\pi,\ 8\pi]$, equation (8) possesses at most one solution of (8) satisfying*

$$u(z) = u(-z) \ \ \text{for} \ \ z \in T.$$

Let $x = \wp(z)$ be the weiestraes $\wp$-function, which is two-fold covering map of T onto $S^2 = \mathbb{C} \cup \{\infty\}$. Set

$$v(x) = u(z) - \log |\wp'(z)|.$$

Then

$$\Delta v(x) + \rho e^v = \Sigma_{j=1}^3 (-2\pi)\delta_{e_j} + \alpha_0 \delta_\infty \ \ \text{in} \ \ S^2,$$

where $e_j = \wp(\frac{w_j}{2})$, and δ_∞ is the Dirac measure at ∞. If $\rho \geq 4\pi$, then $\alpha_0 \geq 0$.

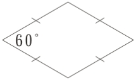
Theorem 11 *Equation (8) with $\rho = 8\pi$ possesses an even solution if and only if the Green function $G(z;\ 0)$ has a non-half period critical point z_0, i.e. $z_0 \neq \frac{w_j}{2}$, $j = 1,\ 2,\ 3$.*

Remark:

1. Since $G(-z;\ 0) = G(z;\ 0)$, $G(z;\ 0)$ always has at least three critical points.
2. If there is a even solution $u(z)$, then there exists a one paramter family $u_\lambda(z)$ of (8) with $\rho = 8\pi$ such that $u(z) = u_{\lambda_0}(z)$ for some λ_0. Thus, Theorem 9 is equivalent to Equation (8) with $\rho = 8\pi$ possesses a solution if and only if the Green function $G(z;\ 0)$ has more than three critical points.

As an application of Theorem 8, we have

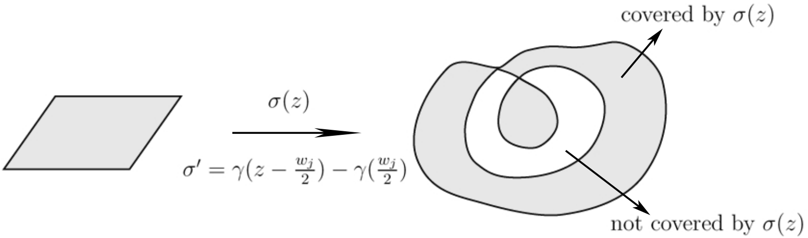
Theorem 11 *For any torus T, the Green function $G(z;\, 0)$ has at most five critical points.*

Example: T =rectangle, then $G(z;\, 0)$ has exactly three critical point $T =$ 

, then $G(z;\, 0)$ has exactly five critical points.

Remark:
1. When G has five critical points. Then all half-periodic $\frac{w_j}{2}$ are saddle points of G.

2. For any half-period point $\frac{w_j}{2}$, we can define



$D(\frac{w_j}{2})$ = the area of components covered by σ minus the area of components not covered by σ.

Question: Does $\det(D^2G(\frac{w_j}{2}; 0))$ have the same sign as $D(\frac{w_j}{2})$?

3. Conjecture: equation (8) possesses at most one solution if $\rho < 8\pi$.

Conclusion: the uniqueness result for the compact Riemann surface of higher genus?