Incompressible limit and non-isentropic fluids

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Introduction

Generalities: Incompressible flows equations justification:

- From compressible Navier–Stokes or Euler equations (shallow-water system)
- Flow velocity field small compared to sound velocity

Limit = incompressible equations.
Correction = acoustic waves, gravity waves.....

Small parameter = Mach number, Froude number
For instance ε = Mach = fluid velocity / sound velocity

- Car: 50 km/h / 120 km/h = 1/20
- Plane: 800 km/h / 1200 km /h = 0.66

velocity motions $<$ 150 km are essentially incompressible

Difference = Noise (waves..)

Introduction

Compressible isentropic Euler equations:

$$\partial_t \rho + \mathrm{div}(\rho u) = 0$$

$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p(\rho) = 0$$

Scaling:

$$u(t,x) = \varepsilon U(\varepsilon\, t, x)$$

gives

$$\partial_t \rho + \mathrm{div}(\rho U) = 0$$

$$\partial_t(\rho U) + \mathrm{div}(\rho U \otimes U) + \frac{\nabla p(\rho)}{\varepsilon^2} = 0$$

Then limit $\varepsilon \longrightarrow 0$ provides

$$\nabla p(\rho) = 0.$$

Thus, using the mass equation, ρ is a constant $\rho = 1 \implies$ divergence free condition $\mathrm{div}\, U = 0$ (U denoted u in the sequel).

Introduction

Wave equation:

$$\psi = \frac{\rho - 1}{\varepsilon}$$

gives

$$\partial_t \psi + \mathrm{div}(\psi u) + \frac{\mathrm{div} u}{\varepsilon} = 0$$

$$\partial_t u + \mathrm{div}(u \otimes u) + h(\psi) + p'(1)\frac{\nabla \psi}{\varepsilon} = 0$$

Combinaison of a wave equation

$$\partial_t \sigma + \mathrm{div} v = 0$$

$$\partial_t v + p'(1)\nabla \sigma = 0$$

with a nonlinear equation.

Time scales:

* $O(1)$: Fluid evolution
* $O(\varepsilon)$: Wave evolution (wave propagation velocity $= 1/\varepsilon$).

Attended result: If we look the incompressible part of u

$\Longrightarrow$ convergence to incompressible Euler

Introduction

Non-exhaustive bibliography:

- S. Klainerman, A. Majda: Existence on a time interval independent on Mach number.
- S. Klainerman, A. Majda: Convergence with well prepared data ($\psi = O(\varepsilon)$).
- S. Ukai: Whole space and waves going to infinity in times $O(\varepsilon)$.
- S. Schochet: incompressible limit, general initial data.
- E. Grenier: Rotating fluids, general initial data.
- I. Gallagher: Oscillating limit parabolic systems.
- B. Desjardins, E. Grenier, P.-L. Lions, N. Masmoudi: Viscous incompressible limit with boundaries, damping.
- B. Desjardins, E. Grenier: Incompressible limit with Strichartz on weak solutions.
- P.–L. Lions, N. Masmoudi Local approach, weak solutions.
- D. Bresch, B. Desjardins, D. Gérard-Varet: Anisotropic compressible rotating fluids in cylinder.

Introduction

Main ideas: Periodic and whole space case

Step 1: wave group

$\mathcal{L}(t)(\sigma_0, v_0)$ group solutions of

$$\partial_t \sigma + \mathrm{div} v = 0$$

$$\partial_t v + \nabla \sigma = 0$$

with initial data (σ_0, v_0).

$\mathcal{L}(t)$ is an isometry from H^s into H^s for

- periodic box
- whole space

Step 2: conjugate process

We conjugate $\mathcal{L}(t)$ posing

$$(\bar{\psi}, \bar{u}) = \mathcal{L}(-t/\varepsilon)(\psi, u)$$

and we get the equation under the form

$$\partial_t(\bar{\psi}, \bar{u}) + \mathcal{L}(-t/\varepsilon) Q(\mathcal{L}(t/\varepsilon)(\bar{\psi}, \bar{u})) = 0$$

Introduction

Step 3: Limit process

$\partial_t(\bar{\rho}, \bar{u})$ is bounded (No problem with compactness in space).

Non-isentropic fluids

The equations read:

$$\partial_t \rho + \mathrm{div}(\rho u) = 0,$$

$$\rho(\partial_t u + u \cdot \nabla u) + \nabla p = 0,$$

with

$$\partial_t S + u \cdot \nabla S = 0.$$

where S entropy, p given by the state law $\rho = R(p, S)$.

Example: $\rho = p^{1/\gamma} e^{-S/\gamma}$.

Change of variable: Let (p, u) then denoting $p = \bar{p}\exp^{\varepsilon q}$, we get

$$a(\partial_t q + u \cdot \nabla q) + \frac{1}{\varepsilon}\mathrm{div} u = 0,$$

$$r(\partial_t u + u \cdot \nabla u) + \frac{1}{\varepsilon}\nabla q = 0$$

$$\partial_t S + u \cdot \nabla S = 0$$

Non-isentropic fluids

Formal limit

From mass and momentum equations:

$\mathrm{div} u = 0$ and $\nabla q = 0$ then

$$\mathrm{div} u = 0,$$

$$r(\partial_t u + u \cdot \nabla u) + \nabla \Pi = 0$$

$$\partial_t S + u \cdot \nabla S = 0$$

with $\rho = R(\bar{p}, S)$ and $r(S)$.

Wave equation:

$$\partial_t(\sigma, v) = \mathcal{A}(\sigma, v)$$

with

$$\mathcal{A} = \begin{pmatrix} 0 & a^{-1}(S)\nabla . \\ r^{-1}(S)\nabla & 0 \end{pmatrix}.$$

which gives

$$\partial_{tt}\sigma - \mathrm{div}(S(t,x)^{-1}\nabla\sigma) = 0.$$

Remark: $\partial_t S$ is bounded but wave equation with variable coefficients.

Non-isentropic fluids

Step 1: Wave equation

$$\partial_t^2 \sigma - \varepsilon^{-2} \mathrm{div}(S(t,x)^{-1} \nabla \sigma) = 0.$$

Let $\mathcal{L}(t)$ the solutions group. We want that $\mathcal{L}(t)$ is bounded uniformly from H^s into H^s.

Energy estimates:

* L^2: Energy gives uniform bound in L^2.
* H^1: ∂_t satisfies a wave equation with unbounded source term with respect to ε.

Spectral decomposition

Problem: Variable coefficients with respect to time !

Problem: multi-eigenvalues possibility !

Problem: Crossing eigenvalues possibility !

$\Longrightarrow$ **bad behavior possibility ....**

Generic results: "for almost all initial data"

Non-isentropic fluids

Two questions:

- Solve equations on some time interval which is independent of Mach number?
- Characterization of the limit when Mach goes to zero?

First question :

- G. Métivier and S. Schochet: Non-isentropic Euler eqs.
- T. Alazard: Full Compressible Navier-Stokes eqs.

Second question:

- G. Métivier and S. Schochet: Whole space and Euler.
- T. Alazard: Exterior domain and Euler; whole space and full CNS eqs.

Non-isentropic fluids

Relies on theorem:

$$\varepsilon^2\partial_t(a^\varepsilon(t,x)\partial_t\phi^\varepsilon) - \mathrm{div}(b^\varepsilon(t,x)\nabla\phi^\varepsilon) = \varepsilon f^\varepsilon(t,x)$$

where

ϕ^ε is bounded in $\mathcal{C}^0([0,T];H^2(\mathcal{R}^d))$, f^ε is bounded in $L^2([0,T];L^2(\mathcal{R}^d))$,

a^ε and b^ε decay to zero at spatial infinity in same similar manner :

$$a^\varepsilon(t,x) \geqslant c, \qquad |a^\varepsilon(t,x) - \underline{\mathrm{a}}| = \mathcal{O}(|x|^{-1-\delta}), \qquad |\nabla a^\varepsilon(t,x)| = \mathcal{O}(|x|^{-2-\delta}),$$

Then ϕ^ε converges strongly to 0 in $L^2_{\mathrm{loc}}([0,T]\times\mathcal{R}^d)$ to $(0,0)$.

Non-isentropic fluids

Singular limit and nonisentropic Euler or NS systems.

- T. ALAZARD. Incompressible limit of the nonisentropic Euler equations with the solid wall boundary conditions. *Adv. Differential Equations* 10 (2005), no. 1, 19–44.
- T. ALAZARD. Low Mach number limit of the full Navier-Stokes equations, *Arch. Ration. Mech. Anal.* 180 (2006), no. 1, 1–73.
- T. ALAZARD. Low Mach number flows and combustion, *SIAM J. Math. Anal.* 38 (2006), no. 4, 1186–1213.
- G. MÉTIVIER, S. SCHOCHET. Averaging theorems for conservative systems and the weakly compressible Euler equations. *J. Differential Equations* 187 (2003), no. 1, 106–183.
- G. MÉTIVIER, S. SCHOCHET. The incompressible limit of the non-isentropic Euler equations. *Arch. Ration. Mech. Anal.* 158 (2001) , no.1, 61–90.
- D. BRESCH, B. DESJARDINS, E. GRENIER, C.K. LIN. Low Mach number limit of viscous polytropic flows: formal asymptotics in the periodic case. *Studies in Applied Math.*, 109, (2002), 125–149.

Non-isentropic fluids

Averaged equation for non-isentropic NS equations:

D. BRESCH, B. DESJARDINS, E. GRENIER, C.K. LIN. Low Mach number limit of viscous polytropic flows: formal asymptotics in the periodic case. *Studies in Applied Math.*, 109, (2002), 125–149.

$$\partial_t \bar\rho + \text{div}\,(\overline{\rho u}) = 0, \qquad \text{div}\,\overline{u} = 0, \qquad \bar\rho\,\bar a = 1,$$

$$\partial_t(\overline{\rho u}) + \text{div}\,(\overline{\rho u} \otimes \overline{u}) + \nabla \bar P - \mu \Delta \overline{u}$$

$$= \sum_{\substack{\ell,m \\ \varphi_\ell = \varphi_m}} \frac{\alpha_\ell^+ \alpha_m^- + \alpha_\ell^- \alpha_m^+}{2} \left(\nabla(\Psi_m \Psi_\ell) - \frac{\overline{a}}{\lambda_\ell^2} \nabla(\nabla\Psi_\ell \cdot \nabla\Psi_m) \right)$$

with (λ_j^2, Ψ_j) denote the eigenvectors of the nonlinear wave equation

$$-\text{div}\,(\overline{a}\nabla\Psi_j) = \lambda_j^2 \Psi_j \quad \text{and} \quad \varphi_j(t) = \int_0^t \lambda_j(s)\,ds.$$

Non-isentropic fluids

The coefficients $\alpha_k^{\sigma_k}$ with $\sigma_k \in \{+,-\}$ denote the components of the acoustic waves on a basis depending on $\{\Psi_j\}_{j\in\mathbb{N}}$. They are governed by the dynamical system

$$\frac{d\alpha_k^{\sigma_k}}{dt} + \frac{\lambda_k^2(\lambda+2\mu)}{2}\alpha_k^{\sigma_k} + \sum_{\substack{\ell \\ \varphi_k=\varphi_\ell}} \mu \frac{\alpha_\ell^{\sigma_k}}{2\lambda_k^2} \int \mathrm{curl}(\bar{a}\nabla\Psi_k)\cdot\mathrm{curl}(\bar{a}\nabla\Psi_\ell)\, dx$$

$$= \sum_{\substack{\ell \\ \lambda_k=\lambda_\ell}} \frac{\alpha_\ell^{\sigma_k}}{2} \int \left\{ \Psi_\ell \partial_t \Psi_k + \frac{\nabla\Psi_\ell}{\lambda_k}\partial_t\left(\frac{\overline{a}\nabla\Psi_k}{\lambda_k}\right)\right\} dx$$

$$+\frac{(\gamma-1)}{4\sqrt{2}} \sum_{\substack{\ell,m,\sigma_\ell,\sigma_m \\ \sigma_\ell\varphi_\ell+\sigma_m\varphi_m=\sigma_k\varphi_k}} i\sigma_k\lambda_k\alpha_\ell^{\sigma_\ell}\alpha_m^{\sigma_m} \int \Psi_k\Psi_m\Psi_\ell\, dx$$

$$-\sum_{\substack{\ell \\ \varphi_\ell=\varphi_k}} \frac{\alpha_\ell^{\sigma_k}}{2\lambda_k^2} \int \overline{a}\ \mathrm{div}\,(\overline{u}\otimes\nabla\Psi_\ell + \nabla\Psi_\ell\otimes\overline{u})\cdot\nabla\Psi_k\, dx$$

$$-\sum_{\substack{\ell,m,\sigma_\ell,\sigma_m \\ \sigma_\ell\varphi_\ell+\sigma_m\varphi_m=\sigma_k\varphi_k}} \frac{i\alpha_\ell^{\sigma_\ell}\alpha_m^{\sigma_m}}{2\sqrt{2}} \frac{1}{\sigma_k\lambda_k\sigma_\ell\lambda_\ell\sigma_m\lambda_m} \int \overline{a}\ \mathrm{div}\,(\overline{a}\nabla\Psi_\ell\otimes\nabla\Psi_m)\cdot\nabla\Psi_k\, dx.$$

Non-isentropic fluids

Some comments:

- Nonhomogeneity $\Longrightarrow$ extra streaming term $\overline{a}\nabla(\nabla\Psi_l \cdot \nabla\Psi_m)$
- Energy exchange between main contribution and waves
- More complex than anelastic limit
- For weak and local process (difficulties): regularity and vanishing properties on $\overline{a}$?

References: anelastic limit with non-vanishing heterogeneity profile

- D. BRESCH, M. GISCLON, C.K. LIN : From degenerate shallow-water system, *M2AN*, Vol. 39, No3, pp. 477-486, (2005).
- E. FEIREISL, J. MALEK, A. NOVOTNY, I. STRASKRABA : bounded domain, damping, *Comm. Partial Diff. Equations*, 157 176, Volume 33 Issue 1 (2008).
- N. MASMOUDI : Local approach. *J. Math Pures et Appliquées*, Volume 88, Issue 3, (2007), 230-240.

Non-isentropic fluids

Non-constant coefficients limit...... Some examples in environmental problems.

Rigid lid approximation for bilayers models

- bi-layers shallow water systems.
- Sedimentation.
- Fluide-Structure interaction.

$\Longrightarrow$ time dependent parameter.

An example.

$$\begin{cases} \partial_t h + \operatorname{div}(hv) = 0, \\ \partial_t(hv) + \operatorname{div}(hv \otimes v) + h\dfrac{\nabla(h + z_b)}{\mathrm{Fr}^2} = 0, \\ \partial_t z_b + \operatorname{div}(q_b(h, v)) = 0 \end{cases}$$

where z_b is the movable bed thickness. Formulas in litterature for q_b : Grass equation, Meyer-Peter and Muller equation, formulas of Nielsen, Fernández Luque and Van Beek.....
– Grass model: Solid transport given by

$$q_b(h, v) = A_g |v|^{m_g} v, \qquad 0 \leqslant m_g \leqslant 3$$

Non-isentropic fluids

Transversality and crossing of eigenvalues.

- R. ABRAHAM, J. ROBBIN. *Transversal mappings and flows.* W.A. Benjamin, New–york, Amsterdam 1967.
- K. UHLENBECK. Generic properties of eigenfunctions. *Amer. J. Math.*, 98, (1976), 1059-1078.
- P. GÉRARD ET F. GOLSE. Averaging Regularity Results for PDEs under Transversality Assumptions. *Comm. on Pure and Appl. Math.* 45, (1992), 126.
- G. MÉTIVIER, S. SCHOCHET. Averaging theorems for conservative sys- tems and the weakly compressible Euler equations. *J. Differential Equations* 187 (2003), no. 1, 106–183.
- G. MÉTIVIER, S. SCHOCHET. The incompressible limit of the non- isentropic Eulerequations. *Arch. Ration. Mech. Anal.* 158 (2001), no.1., 61–90.
- D. BRESCH, B. DESJARDINS, E. GRENIER. Crossing eigenvalues: measure type estimates. *J. Diff. Eqs.* Volume 241, Issue 2, (2007), 207-224.
- D. BRESCH, B. DESJARDINS, E. GRENIER. Measures on double or resonant eigenvalues for linear Schrödinger operator. *J. Functional Anal.* Volume 254, Issue 5, (2008), 1269–1281.

Several papers: C. FERMANIAN, P. GÉRARD, Y. COLIN DE VERDIÈRE.... etc..

Non-isentropic fluids

Spectral decomposition

Let

$$\partial_t^2 \sigma - \varepsilon^{-2} \mathrm{div}(S(x)^{-1} \nabla \sigma) = 0$$

forgetting time dependency

Spectrum:

$-\mathrm{div}(S(x)^{-1} \nabla \cdot)$ is a self-adjoint operator

Eigenvalues λ_j (with eventual multiplicity)

Π_j its corresponding eigenspace and ψ_j orthonormal basis.

Eigenspaces geometry:

Double eigenvalues

$$\Sigma_{j,k} = \Big\{ \lambda_j(S) = \lambda_k(S) \Big\}.$$

In a neighborhood of a double eigenvalue,

$$\Pi_j + \Pi_k$$

is continuous, but not ψ_j , nor ψ_k .

Non-isentropic fluids

Is $\Sigma_{j,k}$ of codimension 2?

A matrix model:

Symetric matrices with eigenvalue at least double are of co-dimension 2 in the symmetric matrices set.

In dimension 2 ....

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

Characteristic polynomial

$$X^2 - (a+c)X + ac - b^2$$

Eigenvalues:

$$\frac{a+c}{2} \pm \frac{\sqrt{(a-c)^2 + b^2}}{2}$$

Then

$$\Sigma_{j,k} = \{b = 0, a = c\}$$

line in a three dimensional space.
The eigenvectors do not depend on $x - \Pi x$ where Π is the projection on $\Sigma_{j,k}$.

Non-isentropic fluids

Is $\Sigma_{j,k}$ of codimension 2?

It seems that all have to be done !!

Question:

$$\mu\Big(S \quad | \quad \lambda_j(S)-\lambda_k(S)| < \varepsilon\Big) \leqslant C\varepsilon^2$$

Difficulties:

- Definition of the measure μ in infinite dimension space ?
- Uniformity with respect to the approximation ?

Let Π_N projection on finite dimension space (Galerkin)
Let

$$\Sigma_{j,k}^{N,\varepsilon} = \{S = \Pi_N(S) \quad | \quad |\lambda_j(S)-\lambda_k(S)| < \varepsilon\}$$

On $\mathbb{R}^N$ the measure of Besov type

$$\mu_N = \otimes_{k=1}^{N} \frac{k^s}{2} 1_{[-1/k^s, 1/k^s]}$$

Non-isentropic fluids

Measure of neighborhoods of $\Sigma_{j,k}$

$$\Sigma_{j,k}^{N,\varepsilon} = \{S = \Pi_N(S) \quad | \quad |\lambda_j(S) - \lambda_k(S)| < \varepsilon\}$$

$$\mu_N = \otimes_{k=1}^{N} \frac{k^s}{2} 1_{[-1/k^s, 1/k^s]}$$

Theorem 0. *Under hypothesis of non degeneracy, there exists a constant C_0 such that*

$$\mu_N(\Sigma_{j,k}^{N,\varepsilon}) \leqslant C_0 \varepsilon^2$$

for all N and all ε.

Proof

Effect of regularity: $\Sigma_{j,k}$ is a graph with respect to the first components $\Pi_N x$.

Remarks:

- Codimension 2 notion "in the measure μ_N sense".
- $\Sigma_{j,k}$ has a null measure too, but what is important is its approximation.

Non-isentropic fluids

Measure of neighborhoods of $\Sigma_{j,k}$

- Approximate diagonalisation
- Ansatz on (ψ_j, ψ_k):

  If $S_0 \in \Sigma_{j,k}$ then $\lambda_j(S_0+S)$ and $\lambda_k(S_0+S)$ are given by

$$\frac{\lambda_j(S_0)+\lambda_k(S_0)}{2} + \frac{1}{2}\Big(\int S|\nabla\psi_j|^2 + \int S|\nabla\psi_k|^2\Big)$$

$$\pm\frac{1}{2}\sqrt{\Big(\int S|\nabla\psi_j|^2 - \int S|\nabla\psi_l|^2\Big)^2 + 4\Big(\int S\nabla\psi_j\nabla\psi_k\Big)^2} + O(|S|^2_{H^s}).$$

  gives informations locally.
- Simple eigenvalues are Lipschitzian

$$\nabla_S\lambda_j(S_0).S = -\int S|\nabla\psi_j|^2.$$

- Eigenvalues cannot be closed too quickly.
- When they are closed ... Above ansatz.

Non-isentropic fluids

Outside $\Sigma_{j,k}$

$$\partial_t^2\sigma - \varepsilon^{-2}\mathrm{div}(S(t,x)^{-1}\nabla\sigma) = 0$$

We decompose

$$\sigma(t) = \sum_j \alpha_j(t)\psi_j(S(t))\exp\Big(\varepsilon^{-2}\int_0^t \lambda_j(S(t))\Big).$$

We get

$$\partial_t\alpha_j = -\Big(\sum_k \alpha_k(t)\nabla\psi_k(S(t)).S'(t) \quad | \quad \psi_j(S(t))\Big).$$

This is correctly bounded from above!

As soon as $S(t)$ avoids double eigenvalues, $\mathcal{L}$ is bounded.

Non-isentropic fluids

Is it possible to avoid $\Sigma_{j,k}$?

Geometry of the problem:

Find initial data which avoid a codimension 2 subset.

Regular flow case in finite dimension

$\Theta(t_1, t_2)$ flow, Σ of codimension 2 to be avoided
We have to evaluate

$$A_\varepsilon = \{x \quad | \quad \exists 0 \leqslant t \leqslant T \quad \Theta(0,t)x \in \Sigma_\varepsilon\}.$$

$$= \cup_t \{x \quad | \quad \Theta(0,t)x \in \Sigma_\varepsilon\}.$$

Two hypothesis:

- Flow with bounded divergence
- Bounded flow

$$\mu(A_\varepsilon) \leqslant C\varepsilon T.$$

Problem: The flow is not regular!!!

Non-isentropic fluids

Limit equation

Well prepared data:

Waves with $O(\varepsilon)$ size. Limit = incompressible non-homogeneous Euler equations

Ill prepared data:

- Waves with $O(1)$ size.
- Limit = Euler with a source term: wave interactions.
- Source term= combinaison of terms involving $\psi_j(S)$ $\Longrightarrow$ singular around to $\Sigma_{j,k}$.

Type equation

ODE of the form

$$\partial_t \phi + Q(\phi) = R\Big(\frac{x - \Pi x}{\|x - \Pi x\|}\Big)$$

with Π projection on a codimension 2 variety.

Non-isentropic fluids

Dimension 2 example

$$\dot{x} = \phi\Big(\frac{x}{|x|}\Big)$$

with ϕ continuous defined from the unit circle to $\mathbb{R}^2$.

Polar coordinates:

$$x(t) = \rho(t)e^{i\theta(t)}$$

with

$$\rho\dot{\theta} = \chi(\theta)$$

$$\dot{\rho} = \psi(\theta)$$

with $\chi(\theta) = \mathcal{I}m\,(\phi(e^{i\theta})e^{-i\theta})$. Change of time gives

$$\dot{\theta} = \chi(\theta)$$

$$\dot{\rho} = \psi(\theta)\rho.$$

Non-isentropic fluids

Discussion

- Possible asymptots: θ with $\chi(\theta) = \theta$.
- Stability depends on χ'.
- Multiple possibility in function of sign of ψ.

Flow:

- The flow is discontinuous: We pass on the left or on the right of the singularity
- or we enter directly in the singularity in finite time.

Divergence:

Through calculation, if A set

$$\mu(\Theta(t)(A)) \leqslant C\mu(A)$$

with C independent on t and on A.

Non-isentropic fluids

Vector field with a homogeneous degre 0 singularity

$$\dot{x} = \phi\Big(x, \frac{x_h}{|x_h|}\Big)$$

with $x_h = (x_1, x_2)$.

- Perturbative arguments with respect to the dimension 2.
- Under geometrical hypothesis: Existence except for a codimension 1 subset.

Non-isentropic fluids

Resonances

$$\Sigma_{j,k,l} = \{S \quad | \quad \lambda_j(S) + \lambda_k(S) = \lambda_l(S)\}.$$

- Heuristically $\Sigma_{j,k,l}$ is of codimension 1.
- Codimension 1 in the measure sense

$$\mu\{S \quad | \quad |\lambda_j(S) + \lambda_k(S) - \lambda_l(S)| < \varepsilon\} \leqslant C\varepsilon.$$

More precisely

Theorem 0. *2 Under non degeneracy hypothesis,*

$$\mu_N\left(\Sigma^{N,\varepsilon}_{j,k,l}\right) \leqslant C\varepsilon.$$

Non-isentropic fluids

Proof of resonance theorem

- Differential calculus

$$d(\lambda_j + \lambda_k - \lambda_l) = \left(|\nabla \psi_j|^2 + |\nabla \psi_k|^2 - |\nabla \psi_l|^2 \right)$$

- The differential does not vanished if

$$|\nabla \psi_j|^2 + |\nabla \psi_k|^2 - |\nabla \psi_l|^2 \neq 0.$$

- Differential belongs to all H^s : eigenvalues vary slowly when perturbate high frequencies.
- Differential depends essentially of the first N components...
- $\Sigma_{j,k,l}$ is a graph with respect to its first N components if N is large enough.

Non-isentropic fluids

In progress: non-homogeneous incompressible limit

- First step: Check that the limit system has a solution for almost all initial data.
- Check that almost all initial data avoids $\Sigma_{j,k}$.
- Conjugate nonhomogeneous incompressible NS equation with $\mathcal{L}$.
- Pass to the limit
- Pass to the limit in the resonances.

Objective: Convergence for almost all initial data......