MARCH 2008

CONVECTION, OPTIMAL TRANSPORT AND COUPLED MONGE-AMPERE SYSTEMS

YANN BRENIER

CNRS, FR 2800, Université de Nice-Sophia-Antipolis,

Web site: http://math1.unice.fr/~ brenier

CONVECTION, OPTIMAL TRANSPORT AND COUPLED MONGE-AMPERE SYSTEMS

1. CONVECTION AND NAVIER-STOKES BOUSSINESQ EQUATIONS

2. HYDROSTATIC BOUSSINESQ EQUATIONS

3. TRANSPORT MONGE-AMPERE SYSTEMS
   (Hoskins' Semi-geostrophic equations and fully non-linear Chemotaxis model)

Université Nice SOPHIA ANTIPOLIS

A NAVIER-STOKES BOUSSINESQ 'NSB' MODEL

Let D be a smooth bounded domain $D \subset R^3$ in which moves an incompressible fluid of velocity $v(t, x)$ at $x \in D$, $t \geq 0$, subject to:

$$\mathbf{NSB}: \quad \epsilon(\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}) + \mathbf{K}\mathbf{v} + \nabla \mathbf{p} = \mathbf{y} \qquad \nabla \cdot \mathbf{v} = 0$$

where $\boxed{\mathbf{K}\mathbf{v} = \alpha \mathbf{v} - \nu \Delta \mathbf{v}}$ with $\alpha \geq 0$, $\epsilon > 0$, $\nu > 0$ and $v = 0$ along ∂D.

The force field y is subject to the advection equation

$$\partial_t \mathbf{y} + (\mathbf{v} \cdot \nabla)\mathbf{y} = \mathbf{G}(\mathbf{x}, \mathbf{y})$$

where G is a given smooth function with bounded derivatives.

CONVECTION THEORY corresponds to the special case

$$\mathbf{G} = 0, \quad \mathbf{y} // \mathbf{e_3}$$

with (usually) an additional diffusion term in the y equation.
Global existence of weak solutions in 3D follows from Leray/Diperna-Lions theory. Global existence of smooth solutions in 2D follows from Hou-Li 2005 and Chae 2006.

Université Nice SOPHIA ANTIPOLIS

THREE LIMITS OF THE NS BOUSSINESQ MODEL

While keeping unchanged

$$\partial_t \mathbf{y} + (\mathbf{v} \cdot \nabla)\mathbf{y} = \mathbf{G}(\mathbf{x}, \mathbf{y}) \qquad \nabla \cdot \mathbf{v} = \mathbf{0}$$

and dropping inertia terms, we consider three possible limit regimes:

STOKES – BOUSSINESQ SB : $\quad \epsilon = \alpha = \mathbf{0}, \ \ \nu = \mathbf{1} \quad \Rightarrow \quad -\Delta \mathbf{v} + \nabla \mathbf{p} = \mathbf{y}$

(the limit $\epsilon \to \mathbf{0}$ can be rigorously justified, YB 2007)

DARCY – BOUSSINESQ DB : $\quad \epsilon = \nu = \mathbf{0}, \ \ \alpha = \mathbf{1} \quad \Rightarrow \quad \mathbf{v} + \nabla \mathbf{p} = \mathbf{y}$

(the limit $\epsilon \to \mathbf{0}$ can be rigorously justified, YB 2007)

HYDROSTATIC – BOUSSINESQ HB : $\quad \epsilon = \nu = \alpha = \mathbf{0} \quad \Rightarrow \quad \nabla \mathbf{p} = \mathbf{y}$

(here the rigorous justification of the limit $\epsilon \to \mathbf{0}$ seems widely open!)

EVOLUTION OF 'OBSERVABLES' IN BOUSSINESQ SYSTEMS

For each suitable test function f, we define the 'observable'

$$\mathbf{t} \to \rho_{\mathbf{f}}(\mathbf{t}) = \int_{\mathbf{D}} \mathbf{f}(\mathbf{y}(\mathbf{t},\mathbf{x}))\mathbf{dx}$$

where y is solution to one of the Boussinesq systems (NSB,SB,DB,HB)

Since $\partial_{\mathbf{t}}\mathbf{y} + (\mathbf{v}\cdot\nabla)\mathbf{y} = \mathbf{G}(\mathbf{x},\mathbf{y})$ where $\nabla\cdot\mathbf{v} = \mathbf{0}, \quad \mathbf{v}//\partial\mathbf{D},$

we get, for each suitable test function f,

$$\frac{\mathbf{d}}{\mathbf{dt}}\int_{\mathbf{D}} \mathbf{f}(\mathbf{y}(\mathbf{t},\mathbf{x}))\mathbf{dx} = \int_{\mathbf{D}} (\nabla\mathbf{f})(\mathbf{y}(\mathbf{t},\mathbf{x}))\cdot\mathbf{G}(\mathbf{x},\mathbf{y}(\mathbf{t},\mathbf{x}))\mathbf{dx}$$

EVOLUTION OF 'OBSERVABLES' IN THE HYDROSTATIC CASE

The Hydrostatic Boussinesq 'HB' model
just requires y to be a gradient: $\mathbf{y = \nabla p}$.

A remarkable fact : y is COMPLETELY DETERMINED by its observables $\mathbf{t \to \rho_f(t) = \int_D f(y(t,x))dx}$ (for all suitable test functions f)

under the following A PRIORI CONVEXITY ASSUMPTION

$\mathbf{p(t,x)}$ is a CONVEX function of $\mathbf{x \in D}$ (D being supposed to be convex)

NB: This is a typical result of OPTIMAL TRANSPORT THEORY:
YB, C. R. Acad. Sci. Paris 1987 and CPAM 1991 Smith and Knott, J. Optim. Theory Appl. 1987 Caffarelli, J. AMS 1992 and Ann. of Math.1996, Villani, Topics in optimal transportation, AMS, 2003, see also reviews and lecture notes and many other papers and books.

HYDROSTATIC BOUSSINESQ: A GLOBAL EXISTENCE RESULT

YB 2007, also see G. Loeper, SIAM J. Math. Anal. 2006

THEOREM

Assume $G(x, y)$ to be a smooth function with bounded first derivatives.
Let C be the convex cone of all maps $y \in L^2(D, R^3)$
such that $y(x) = \nabla p(x)$ a.e. in D for some CONVEX convex lsc function p.

Then, for each $y_0 \in C$, there is $(t \to y(t, \cdot)) \in C^0(R_+, L^2(D, R^3))$
valued in the cone C such that $y(t = 0, \cdot) = y_0$ and

$$\frac{d}{dt}\int_D f(y(t,x))dx = \int_D (\nabla f)(y(t,x)) \cdot G(x, y(t,x))dx,$$

(for all test functions f) which we call a
SOLUTION WITH CONVEX POTENTIAL TO THE HB SYSTEM

$$\partial_t y + (v \cdot \nabla)y = G(x, y), \ \ \nabla \cdot v = 0, \ \ y = \nabla p$$

INTERPRETATION OF THE HB SYSTEM AS A COUPLED MONGE-AMPERE-TRANSPORT SYSTEM

Under the **POTENTIAL CONVEXITY** assumption, the HB system

$$\mathbf{HB:}\quad \partial_t y + (v\cdot\nabla)y = G(x,y), \qquad y = \nabla p, \qquad \nabla\cdot v = 0$$

is (formally) equivalent to the transport-Monge-Ampère system

$$\mathbf{TMA:}\quad \partial_t \rho + \nabla\cdot(\rho w) = 0, \quad w = G(\nabla p^*(t,x),x), \quad \rho = \det(D^2 p^*(t,x))$$

where p^* is the **LEGENDRE-FENCHEL** transform

$$p^*(t,x) = \sup_{\tilde{x}\in D} \quad x\cdot\tilde{x} - p(t,\tilde{x})$$

Indeed, using the change of variable $x = \nabla p(t,\tilde{x}) \iff \tilde{x} = \nabla p^*(t,x)$,

$$\frac{d}{dt}\int f(x)\det(D^2p^*(t,x))dx - \int \nabla f(x)\cdot G(\nabla p^*(t,x),x)\det(D^2p^*(t,x))dx$$

$$= \frac{d}{dt}\int_D f(y(t,\tilde{x}))d\tilde{x} - \int_D (\nabla f)(y(t,\tilde{x}))\cdot G(\tilde{x},y(t,\tilde{x}))d\tilde{x} = 0$$

EXAMPLES OF TRANSPORT-MONGE-AMPERE SYSTEMS 1

Example 1: Setting $\mathbf{G}(\mathbf{x}, \mathbf{y}) = (\mathbf{y_2} - \mathbf{x_2}, \mathbf{x_1} - \mathbf{y_1}, \mathbf{0})$ we recover Hoskins' **SEMI-GEOSTROPHIC** equations.

Then, the **CONVEXITY PRINCIPLE** for the **HB** system exactly corresponds to the **CULLEN-PURSER PRINCIPLE**.

cf. Cullen-Norbury-Purser 1991, Benamou-Brenier 1998, Cullen-Gangbo 2001, Loeper 2006.

Example 2: With $\mathbf{G}(\mathbf{x}, \mathbf{y}) = \frac{\mathbf{y}-\mathbf{x}}{\beta}$ where $\beta > 0$ is a constant, Setting

$$\boxed{\nabla \mathbf{p}^*(\mathbf{t}, \mathbf{x}) = \mathbf{x} - \beta \nabla \psi(\mathbf{t}, \mathbf{x})} \text{ we get}$$

$$\boxed{\mathbf{TMA}: \ \partial_\mathbf{t} \rho + \nabla \cdot (\rho \mathbf{w}) = \mathbf{0}, \quad \mathbf{w} = \nabla \psi(\mathbf{t}, \mathbf{x}), \quad \rho = \det(\mathbf{I} - \beta \mathbf{D}^2 \psi(\mathbf{t}, \mathbf{x}))}$$

EXAMPLES OF TRANSPORT-MONGE-AMPERE SYSTEMS 2

The resulting system can be seen as a FULLY NON-LINEAR CHEMOTAXIS model.

Indeed, Assuming $|\beta| << 1$, the MONGE-AMPERE becomes

$$\rho = \mathbf{det}(\mathbf{I} - \beta \mathbf{D^2}\psi(\mathbf{t}, \mathbf{x})) = \mathbf{1} - \beta \boldsymbol{\Delta}\psi + \mathbf{O}(\beta^{\mathbf{2}})$$

which approximates the CHEMOTAXIS model (without viscosity) considered by Jäger and Luckhaus Trans. AMS 1992:

$$\partial_{\mathbf{t}}\rho + \nabla \cdot (\rho \mathbf{w}) = \mathbf{0}, \quad \mathbf{w} = \nabla\psi(\mathbf{t}, \mathbf{x}), \quad \rho = \mathbf{1} - \beta \boldsymbol{\Delta}\psi(\mathbf{t}, \mathbf{x}))$$

EXAMPLES OF TRANSPORT-MONGE-AMPERE SYSTEMS 3

In one space variable the approximation is exact:

$$\partial_t \rho + \partial_x(\rho \mathbf{w}) = \mathbf{0}, \quad \rho = \mathbf{1} - \beta \partial_x \mathbf{w}$$

In that case, the system can be reduced to the inviscid BURGERS equation

$$\partial_t \mathbf{w} + \partial_x(\mathbf{w}^2/\mathbf{2}) = \frac{\mathbf{w}}{\beta}$$

and the Kruzhkov-Oleinik ENTROPY CONDITION condition exactly fits with the CONVEXITY PRINCIPLE we used for the HB system.