Control of Schrödinger equations

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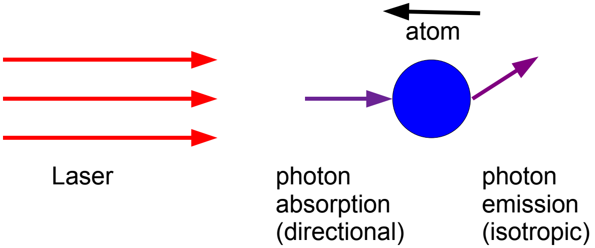
What is quantum control ?

Goal : atoms and small molecules manipulation : control of their orientation, motion, speed, energy level

Physics theory : quantum mechanics
(10^{-12} - 10^{-15} s., 10^{-6} - 10^{-10} m.)

The laser, a good tool : well understood interaction, precision

First application : laser atom cooling → trapping (Nobel Prize 1997, Cohen-Tanoudji)



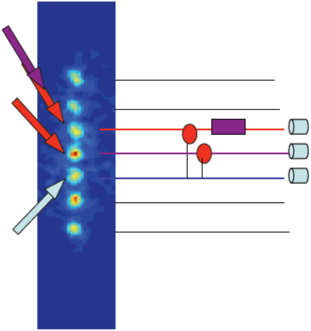
conservation of the linear momentum
⇒ atom cooling

Link : control of the energy level / control of the motion

Second application : quantum computer

Principle : using the qubit (= state of a quantum particle controlled by laser) instead of the bit (= 0 or 1) as information unit

⇒ high performance computing



Des impulsions laser appliqués séquentiellement aux ions de la chaîne réalisent des portes à un bit et des portes à deux bits. La détection par fluorescence (éventuellement précédée par une rotation du bit) extrait l'information du système.

Source : S.Haroche, Collège de France, 2006

needs a good understanding of laser control of atoms energy levels

The studied model

A particle in an infinite square potential well, subjected to a uniform electric field $u : t \mapsto u(t) \in \mathbb{R}^N$.

$$\begin{array}{rccccc} \psi : & [0,T] & \times & \Omega & \to & \mathbb{C} \\ & (t & , & x) & \mapsto & \psi(t,x) \end{array} \qquad \int_\Omega |\psi(t,x)|^2 dx = 1$$

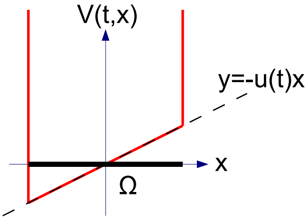
$$\begin{cases} i\frac{\partial \psi}{\partial t}(t,x) = -\Delta\psi(t,x) - \langle u(t), \mu(x)\rangle \psi(t,x) \text{ , } x \in \Omega \\ \psi(t,x) = 0 \text{ , } x \in \partial\Omega. \end{cases}$$

$\mu : \Omega \to \mathbb{R}^N$: dipolar moment

1D example :

$$V(t,x) = -\langle u(t), \mu(x) \rangle$$

$$\mu(x) = x$$

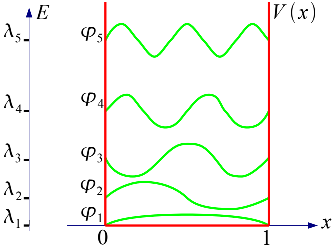


Questions answered in this presentation

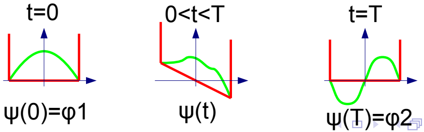
$$\begin{cases} i\frac{\partial\psi}{\partial t}(t,x) = -\Delta\psi(t,x) - \langle u(t),\mu(x)\rangle\psi(t,x) \;,\; x\in\Omega \\ \psi(t,x)=0 \;,\; x\in\partial\Omega. \end{cases}$$

state : ψ **control :** u

Local controllability around eigenstates ?
In arbitrarily small time ?
Controllability between eigenstates ?



Qu : Find $T>0$ and $u:(0,T)\to\mathbb{R}$ s.t.



Brief bibliography
.1. Finite dimensional systems : Lie rank condition

$$\frac{dx}{dt} = f(x,u) \ , \ x \in \mathbb{R}^n \ , \ u \in \mathbb{R}^m.$$

f analytic + local controllability around the equilibrium (x_e, u_e)
$\Rightarrow \left\{ g(x_e); g \in \text{Lie}\left(\frac{\partial^\alpha f}{\partial u^\alpha}(., u_e) \right) \right\} = \mathbb{R}^n.$

$$i\frac{d\psi}{dt} = H_0\psi + u(t)H_1\psi \quad \psi \in \mathbb{C}^N, H_0, H_1 \text{ hermitian}$$

Theorem (Albertini, D'Alessandro, 01) : Controllability
$\Leftrightarrow$ Lie algebra generated by H_0 and H_1 conjugated
- su(N), if N is odd,
- su(N) or sp(N/2), if N is even.

Agrachev, Altafini, Boscain, Chambrion, Charlot, Gauthier, Jauslin, Sigalotti ...
Mabuchi, Maday, Mirrahimi, Rabitz, Rouchon, Salomon, Turinici

Brief bibliography
.2. Iterated Lie brackets in infinite dimension

Sometimes, they give the right intuition.

Harmonic Oscillator : $i\dot{\psi} = -\psi'' + x^2\psi - u(t)x\psi, x \in \mathbb{R}$

$f_0(\psi) := -\psi'' + x^2\psi,$ $\qquad f_1(\psi) := x\psi$
$[f_0, f_1] = -2d/dx,$ $\quad [f_0, [f_0, f_1]] = 4f_1,$ $\quad [f_1, [f_0, f_1]] = 2Id,$
$\Rightarrow$ $\dim[\text{Lie}(f_0, f_1)] = 4.$ Mirrahimi, Rouchon (04)

Butkovskiy, Samoilenko : non controllability

$$\psi(t,x) = \phi(t, x-\eta)e^{i[\dot{\eta}(x-\eta)+\sigma]} \text{ où } i\dot{\phi} = -\phi'' + x^2\phi$$

Navier-Stokes, Euler : Agrachev-Sarychev, Shirikyan

Sometimes they give no intuition...

Example : $i\frac{\partial \psi}{\partial t} = -\psi'' - u(t)x^2\psi, \quad \psi(t,0) = \psi(t,1) = 0$

$D(f_0) := H^2 \cap H^1_0((0,1),\mathbb{C}), \qquad f_0(\psi) := -\psi'', \qquad f_1 := x^2\psi$

Formal computing, for $\psi \in D(f_0)$

$[f_0, f_1](\psi) = -4x\psi' - 2\psi, \qquad [f_1,[f_0,f_1]](\psi) = 8f_1(\psi),$

extension : $f_0(\xi) := -\xi'' + \xi(0)\delta_0' - \xi(1)\delta_1'$

$[f_0,[f_0,f_1]](\psi) = -8f_0(\psi) + 4\psi'(1)\delta_1',$
$[f_0,[f_0,[f_0,f_1]]](\psi) = ?$

$\Rightarrow$ Lie$(f_0,f_1) = ?$.

However this system is loc. controllable around the eigenstates.

A Brief bibliography
.3. Schrödinger PDEs

Linear syst : $i\dot{\psi} = -\Delta\psi + u(t,x)1_\omega, x \in \Omega$

Machtyngier-Zuazua : multiplicators (90)
Lasieka-Triggiani-Zhang : Carleman (92)
Lebeau : microlocal analysis, under GCC (92)
Burq : without GCC (93)
Ramdani, Takahashi, Tenenbaum, Tucsnak : Ω=square, ω arbitrarily small (05), boundary control
Baudouin-Puel : Carleman (03), inverse problem ...

Bilinear syst :
Mirrahimi-Rouchon : harmonic oscillator
Baudouin-Kavian-Puel-Salomon : NL Hartree, optimal control
Cancès-Le Bris-Pilot
Turinici, Ilner-Lange-Teissman : non controllability

Steps of the presentation

$$\begin{cases} i\dot{\psi}(t,x) = -\Delta\psi(t,x) - < u(t), \mu(x) > \psi(t,x), \; x \in \Omega, \\ \psi(t,.) = 0 \text{ on } \partial\Omega. \end{cases}$$

1st step : Local controllability around 1D eigenstates, non pathological case (μ)

2nd step : 1st pathological example

3rd step : Particle in a moving potential well,
2nd pathological example

4th step : Controllability between 1D eigenstates

5th step : One particle in 2D, 3D

Local controllability around 1D eigenstates,

non pathological case (μ)

A previous negative result for this quantum system

$$(\Sigma)\begin{cases} i\dot{\psi} = -\psi'' - u(t)\mu(x)\psi, & x \in (0,1) \\ \psi(t,0) = \psi(t,1) = 0 \end{cases}$$

Ball, Marsden, Slemrod (82) :

$$\dot{w}(t) = \mathcal{A}w(t) + p(t)\mathcal{B}w(t), \quad w \in \mathcal{X}$$

- not controllable :

  $\{w(t;p,w_0); t \geqslant 0, p \in L^r_{loc}(\mathbb{R},\mathbb{R}), r > 1\}$ has empty interior in $\mathcal{X}$

- approximate controllability

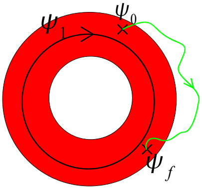
Turinici (00) : non controllability of (Σ) in $H^2 \cap H^1_0((0,1),\mathbb{C})$ with L^2 controls, $\forall \mu$.

A positive result : local controllability

$$(\Sigma) \qquad i\dot{\psi} = -\psi'' - u(t)\mu(x)\psi, \qquad \psi(t,0) = \psi(t,1) = 0.$$

Ground state for $u \equiv 0$:

$$\psi_1(t,x) := \varphi_1(x)e^{-i\lambda_1 t}$$



Theorem : Let $T > 0$, $\epsilon > 0$, $\mu \in W^{6,\infty}((0,1),\mathbb{R})$. We assume

$$\exists c_1, c_2 > 0, \qquad \frac{c_1}{k^3} \leqslant |\langle \mu\varphi_1, \varphi_k \rangle| \leqslant \frac{c_2}{k^3}, \qquad \forall k \in \mathbb{N}^*.(*)$$

There exists $\eta > 0$ such that, $\forall \psi_0, \psi_f \in \mathcal{S} \cap H^{5+\epsilon}_{(0)}((0,1),\mathbb{C})$ with

$$\|\psi_0 - \psi_1(0)\|_{H^{5+\epsilon}} < \eta, \quad \|\psi_f - \psi_1(T)\|_{H^{5+\epsilon}} < \eta,$$

there exists a trajectory (ψ, u) of (Σ) on $[0, T]$ such that $\psi(0) = \psi_0$, $\psi(T) = \psi_f$, $u \in H^1_0((0,T),\mathbb{R})$ and $(\psi_0, \psi_f) \mapsto u$ is continuous.

Some remarks about the assumption (*)

$$\exists c_1, c_2 > 0, \qquad \frac{c_1}{k^3} \leqslant |\langle \mu\varphi_1, \varphi_k \rangle| \leqslant \frac{c_2}{k^3}, \qquad \forall k \in \mathbb{N}^*. (*)$$

$$\langle \mu\varphi_1, \varphi_k \rangle = \int_0^1 \mu(x) \sin(\pi x) \sin(k\pi x) dx$$

Generic with respect to $\mu \in W^{6,\infty}((0,1), \mathbb{R})$: $\mu'(1) \pm \mu'(0) \neq 0$

$\Rightarrow$ very general result

(*) not satisfied when

- $\langle \mu\varphi_1, \varphi_K \rangle = 0$ for some $K \in \mathbb{N}^*$
- or μ is symmetric wrt $x = 1/2$

$\Rightarrow$ pathological cases (steps 2 et 3)

Classical approach for the proof

Let (ψ^*, u^*) be a trajectory of (Σ).

Linearized system around (ψ^*, u^*) controllable in time T.

$\Downarrow$ (often)

Nonlinear system locally controllable around $(\psi^*(0), \psi^*(T))$ in time T.

Proof : Inverse mapping theorem

$$\Theta_T : (\psi_0, u) \mapsto (\psi(0), \psi(T))$$

where ψ solves (Σ) with control u and initial condition ψ_0.

Steps of the proof :

$$\exists c_1, c_2 > 0, \qquad \frac{c_1}{k^3} \leqslant |\langle \mu\varphi_1, \varphi_k\rangle| \leqslant \frac{c_2}{k^3}, \qquad \forall k \in \mathbb{N}^*.$$

(1) Controllability of the linearized system

(2) The spaces in which this controllability holds do not allow the use of the inverse mapping theorem.

(3) Application of the Nash-Moser theorem

Controllability of the linearized system in $H^3_{(0)}((0,1),\mathbb{C})$ with $L^2((0,T),\mathbb{R})$ controls, $\forall T>0$

$$i\dot{\Psi} = \Psi'' - v(t)\mu(x)\psi_1, \qquad \Psi(t,0)=\Psi(t,1)=0, \qquad \Psi(0)=0$$

$$\Psi(t,x) = \sum_{k=1}^{\infty} x_k(t)\varphi_k(x)$$

$$x_k(T) = i\langle \mu\varphi_1, \varphi_k\rangle \int_0^T v(t) e^{i(\lambda_k-\lambda_1)t} dt e^{-i\lambda_k T}.$$

$$\Psi(T)=\Psi_f \qquad \Leftrightarrow \qquad \int_0^T v(t)e^{i(\lambda_k-\lambda_1)t} = \frac{\langle \Psi_f, \varphi_k\rangle e^{i\lambda_k T}}{i\langle \mu\varphi_1, \varphi_k\rangle}$$

Ingham inequality, Haraux (89) $\qquad \lambda_k - \lambda_1 = (k^2-1)\pi^2$

The IMT cannot be used

$$\begin{array}{rccc} \Theta : & H^3_{(0)} \times H^1 & \rightarrow & H^3_{(0)} \times H^3_{(0)} \\ & (\psi_0, u) & \mapsto & (\psi(0), \psi(T)) \end{array}$$

Classical situation :
1) $\Theta \in C^1(H^3_{(0)} \times H^1, H^3_{(0)} \times H^3_{(0)})$
2) $d\Theta(\varphi_1, 0)$ is surjective from $H^3_{(0)} \times H^1$ to $H^3_{(0)} \times H^3_{(0)}$

Here, loss of regularity : $d\Theta(\varphi_1, 0)^{-1} : H^3_{(0)} \times H^3_{(0)} \rightarrow H^3_{(0)} \times L^2$

We build an L^2 control, it should be H^1, but it is impossible.

$\rightarrow$ Nash-Moser theorem

The Nash-Moser theorem (Hörmander, 85)

Inverse mapping	**Nash-Moser**
$\Theta \in C^1(E, F)$	$\Theta \in C^2(E_a, F_a), \forall a$, E_a, F_a Sobolev
$\exists d\Theta(0)^{-1} : F \to E$	$\exists d\Theta(x)^{-1} : F_b \to E_a, \forall x, b > a$ + tame estimates
	$\lvert d^2\Theta(x)\rvert \leqslant \ldots$
$\exists \Theta^{-1} : F \to E$	$\exists \Theta^{-1} : F_{b+\epsilon} \to E_a$
$x_{n+1} = x_n - d\Theta(0)^{-1}[\Theta(x_n) - y]$	$x_{n+1} = x_n - R_n d\Theta(x_n)^{-1}[\Theta(x_n) - y]$

Main difficulty : controllability of an infinite number of linearized syst + estimates

(1) Building of an admissible $d\Theta(0)^{-1}$:
we search $w \in H^2 \cap H^1_0((0,T),\mathbb{R})$ solution of

$$\int_0^T w(t) e^{i\omega_k t} dt = d_k$$

with $\|w\|_{L^2} \leqslant C\|d\|_{l^2}$ and $\|w\|_{H^2} \leqslant C\|d\|_{h^4}$.

Candidate for $T = 2/\pi$, $\omega_k := (k^2-1)\pi^2$:

$$w(t) := 2\Re\left(\sum_{k=1}^{\infty} d_k e^{-i\omega_k t}(1 - cos(\pi^2 t))\right)$$

(2) Building of an admissible $d\Theta(x)^{-1}$: close linear maps, in a sense adapted to tame estimates.

Local controllability around 1D eigenstates

1st pathological example

Studied situation and result

$$(\Sigma) : i\dot{\psi} = -\psi'' - u(t)\mu(x)\psi \ , \ x \in (0,1)$$

Assumption : $\langle \mu\varphi_1, \varphi_1 \rangle = 0$ and

$$\exists c_1, c_2 > 0, \qquad \frac{c_1}{k^3} \leqslant |\langle \mu\varphi_1, \varphi_k \rangle| \leqslant \frac{c_2}{k^3}, \qquad \forall k \geqslant 2$$

Theorem : There exists $T_{min} \in (0, 2/\pi)$ such that,

- $\forall T > T_{min}$, (Σ) is locally controllable in time T around $(\psi_1, u \equiv 0)$, in $H^{5+}_{(0)}((0,1), \mathbb{C})$, with $H^1_0((0,T), \mathbb{R})$ controls
- $\forall T < T_{min}$, there exists $\epsilon > 0$ such that, $\forall u \in L^2((0,T), \mathbb{R})$ with $\|u\|_{L^2} < \epsilon$ then

$$\psi(T) \neq (\sqrt{1-\delta^2} + i\delta)\psi_1(T), \forall \delta > 0.$$

Proof of the controllability $\forall T > T_{min}$

$$(\Sigma) : i\dot{\psi} = -\psi'' - u(t)\mu(x)\psi \ , \ x \in (0,1)$$

Assumption : $\langle \mu\varphi_1, \varphi_1 \rangle = 0$ and

$$\exists c_1, c_2 > 0, \qquad \frac{c_1}{k^3} \leqslant |\langle \mu\varphi_1, \varphi_k \rangle| \leqslant \frac{c_2}{k^3}, \qquad \forall k \geqslant 2$$

(1) Local controllability "up to codimension one "of the nonlinear system around $(\psi_1, u \equiv 0)$

(2) Use of the 2nd order term to move in the missed directions

(3) Conclusion thanks to the Brouwer fixed point theorem

KdV : Cerpa, Coron, Crépeau,

Proof the non controllability $\forall T < T_{min}$

(1) There exists $T_{min} \in (0, 2/\pi)$ for doing the missed motions with the second order term.

(2) For $T < T_{min}$ and with small controls, the solution of the nonlinear syst. stays close enough to (ground state + 1st order + 2nd order) for the same motion to be impossible.

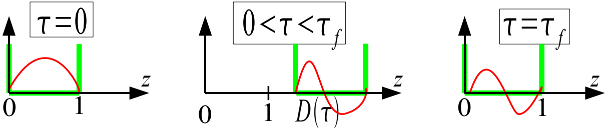
Particle in a moving potential well

2nd pathological example

The studied equation

$$i\frac{\partial \phi}{\partial \tau} = -\frac{\partial^2 \phi}{\partial z^2} + V(z - D(\tau))\phi$$

state : ϕ **control :** $\ddot{D} : [0, T] \to \mathbb{R}$



$x := z - D(t)$ $u := -\ddot{D}/2$ $\phi(t,z) := \psi(t,x)e^{\theta(t,x)}$

$\theta := i(\frac{1}{2} < x, \dot{D} > + \frac{1}{4}\int_0^t |\dot{D}|^2)$

$$i\frac{\partial \psi}{\partial t} = -\psi'' - u(x - 1/2)\psi \ , \ x \in (0,1)$$

Pathology and result

$$(\Sigma)\begin{cases} i\frac{\partial\psi}{\partial t} = -\psi'' - u(x-1/2)\psi \ , \ x\in(0,1) \\ \psi(t,0)=\psi(t,1)=0 \end{cases}$$

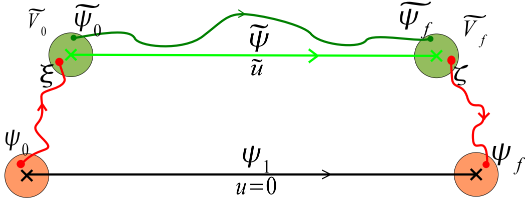
$$\langle \mu\varphi_1, \varphi_k\rangle = \frac{-4k(1+(-1)^k)}{\pi^2(k^2-1)^2}.$$

Theorem :

- There exists $T>0$ such that (Σ) is locally controllable in time T, in $H^{5+}_{(0)}((0,1),\mathbb{C})$, with $H^1_0((0,T),\mathbb{R})$ controls.
- There exists $T_{min}>0$ such that, $\forall T<T_{min}$, there exists $\epsilon>0$ such that, $\forall u\in L^2((0,T),\mathbb{R})$ with $\|u\|_{L^2}<\epsilon$ then

$$\psi(T)\neq(\sqrt{1-\delta^2}+i\delta)\psi_1(T), \forall\delta>0.$$

Strategy : the return method



1) Find a trajectory $(\widetilde{\psi}, \widetilde{u})$ of (Σ) such that the linearized system around $(\widetilde{\psi}, \widetilde{u})$ is controllable.

2) Build neighborhoods and trajectories.

The return method

Stabilization : Coron (92)

Controllability of PDEs :

Coron : Euler 2D (93), Navier-Stokes (96), Shallow water (02)

Horsin : Burgers (98)

Fursikov, Imanuvilov : Navier-Stokes, Boussinesq (99)

Glass : Euler (00,05), Vlasov Poisson (03), Euler isentropique (07), Camassa-Holm (07)

Glass, Guerrero : Burgers (07)

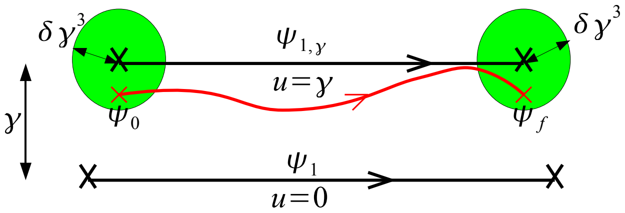
The return method, 1st step

$$i\dot{\psi} = \psi'' - u(t)(x - 1/2)\psi \; , \; x \in (0,1)$$

Ground state for $u \equiv \gamma$: $\psi_{1,\gamma}(t,x) := \varphi_{1,\gamma}(x)e^{-i\lambda_{1,\gamma}t}$

$$-\varphi''_{1,\gamma} - \gamma(x - 1/2)\varphi_{1,\gamma} = \lambda_{1,\gamma}\varphi_{1,\gamma}$$

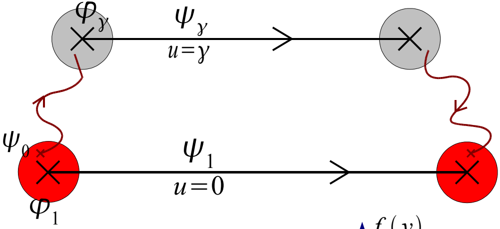
Local controllability in any time $T > 0$



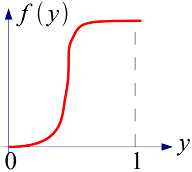
Remark : $\delta\gamma^3 << \gamma$

proof : as in the non pathological case : $\langle \mu\varphi_{1,\gamma}, \varphi_{k,\gamma} \rangle \neq 0, \forall k$

The return method, 2nd step : adiabatic transformations



$$\begin{cases} i\frac{\partial \psi}{\partial t} = -\psi'' - \gamma f(\epsilon t)\psi, \\ \psi(0) = \varphi_1 \end{cases}$$



$\exists C > 0$ such that $\forall k > 0$, $\forall \gamma > 0, \forall \epsilon > 0,$

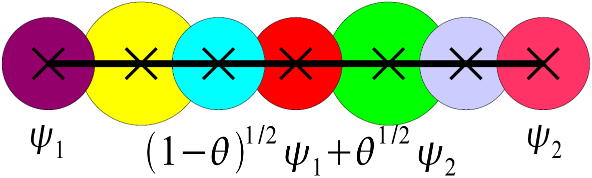
$$\|\psi(1/\epsilon) - \varphi_{1,\gamma} e^{i\phi(\gamma,\epsilon)}\|_{L^2((0,1),\mathbb{R})} \leqslant \gamma C^k \epsilon^k$$

Coron : Shallow water (02)
The adiabatic theorem : a classical tool in quantum mechanics

4th step :

Controllability between eigenstates in 1D

Strategy of the proof



5th step :

Particle in 2D, 3D

Controllability of the linearized systems in 2D-3D ?

A weaker notion of controllability : spectral controllability

$$\psi(0) = \sum_{\text{finite}} a_k \varphi_k \quad \longrightarrow \quad \psi(T) = \sum_{\text{finite}} b_k \varphi_k$$

2D :

- $\langle \mu\varphi_1, \varphi_k \rangle \neq 0, \forall k \in \mathbb{N}^*$
- $\mathrm{Sp}(\Delta_\Omega^D)$ simple and $\sharp\{\lambda_k \in [0,t]\} = dt + O(t^\alpha)$

$\Rightarrow$ $\forall T > 2\pi d$ spectral controllability
$\forall T < 2\pi d$ no spectral controllability

3D : no spectral controllability

proof : Haraux-Jaffard

$\Rightarrow$ controllability of 2D linearized systems, but in abstract spaces.

Conclusion

	controllability of the lin. syst around the ES	spectral controllability of the lin. syst. around the ES	local controllability of the NL syst. around the ES	controllability between ES of the NL syst. linéaire
$1D$	**yes**/**no**	**yes**/**no**	**yes** : H^{5+} with H^1_0controls $T > T_*$ **no** : H^2 with L^2controls	**yes** with H^1_0controls
$2D$	**yes** generic $T > T_*$ abst. spaces	**yes** generic $T > T_*$	Open Pb	Open
$3D$	**no**	**no**	Open Pb	Open Pb

Extension of these methods to other situations

Quantum particle in a variable domain

$$i\dot{\psi} = -\psi'' + (\dot{u} - u^2)x^2\psi \ , \ \psi(t,0) = \psi(t,1) = 0$$

Beam equation

$$u_{tt} + u_{xxxx} + p(t)u_{xx} = 0 \ , \ u = u_x = 0 \text{ at } x = 0,1$$

Simultaneous control (in progress with Mirrahimi)

$$i\xi_k = -\xi_k'' - u(t)x\xi_k, \ \ x \in I, \ \ 1 \leqslant k \leqslant M$$

Local controllability of $(\xi_1, ..., \xi_M)$ around $(\psi_1, ..., \psi_M)$.

Computation of controls

$$i\dot{\psi} = -\psi'' - u(t)\mu(x)\psi \ , \ 0 < x < 1$$

Previous approach $\Rightarrow$ algorithmically computable controls
But difficult computations

For easily computable controls : **Lyapunov approach**

$$L(\psi) := \|\psi - \psi_1\|_{L^2}^2 = 2(1 - \Re\langle\psi, \psi_1\rangle)$$

$$\frac{dL}{dt}\Big[\psi(t)\Big] = 2u(t)\Im\langle\mu\psi(t), \psi_1(t)\rangle$$

$$u(\psi) := -\Im\langle\mu\psi(t), \psi_1(t)\rangle$$

Qu : Do we have $L[\psi(t)] \to 0$ for the closed loop syst. ?
Difficulties : L^2 compactness of the trajectories ? (LaSalle princ.)

Joint work with Mirrahimi : explicit feedback law that realize

$$\limsup_{t\to+\infty} L[\psi(t)] < \epsilon$$