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On the Generalized Dirichlet Problem for Viscous Hamilton-Jacobi Equations

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Question : is there a difficulty to solve the initial-boundary value problem (Dirichlet problem)

$$\begin{aligned} u_t - \Delta u + |Du|^p &= 0 \quad \text{in } \Omega \times (0, +\infty) \\ u(x,0) &= u_0(x) \quad \text{on } \overline{\Omega} \\ u(x,t) &= \varphi(x,t) \quad \text{on } \partial\Omega \times (0, +\infty) \end{aligned}$$

in the case where Ω is a smooth, bounded domain of $\mathbb{R}^N$, $p > 0$ and u_0, φ are continuous functions satisfying the compatibility condition

$$u_0(x) = \varphi(x,0) \quad \text{on } \partial\Omega \ ?$$

Answer 1 :

• If $0 < p < 2$, NO PROBLEM : take your favorite reference on parabolic equations and just follows the theory !

• If $p = 2$, a limiting case, the change $v = -\exp(-u)$ reduces this equation to the Heat equation : again NO PROBLEM.

Answer 2 :

• **If $p > 2$, standard theory does not apply because of the superquadratic growth in the gradient.**

Typical results on the solvability of the Dirichlet problem can be summarized as follows :

the solution exists on some time interval $[0, T)$ and when $t \longrightarrow T$, the gradient of the solution blows up on the boundary while the solution itself remains bounded.

This singularity is a difficulty to extend the solution past T and even one may think that no solution can be defined past T (cf., for example, Fila & Lieberman and Souplet)

Question : can we solve the Dirichlet problem past T or not ?

The answer is YES and NO...

depending on what you call "solving the Dirichlet problem"...

Now we examine the problem by making some remarks.

Remark 1 : As long as the solution exists (or formally), it satisfies the uniform L^∞ estimates

$$||u||_\infty \le \max(||u_0||_\infty, ||\varphi||_\infty)$$

where the L^∞-norm is taken for $x \in \overline{\Omega}$ and $t \in [0, T']$ where T' can be any time if you argue formally.

This is a priori a good point to extend the solution past T...

Remark 2 : in dimension 1, we have "maximal solutions" (for a given initial data).

Indeed, we can solve the problem

$$-\chi''(x) + |\chi'(x)|^p + c^p = 0 \quad \text{in } (-1,1)$$

with $\chi'(x) \to \pm\infty$ when $x \to \pm 1$. Here c is a positive constant which has to be determined.

Since the solution χ is expected to be even, we integrate once and we choose $\chi'(0) = 0$: the solution χ satisfies the equation

$$c^{1-p} \int_0^{\frac{\chi'(x)}{c}} \frac{ds}{|s|^p + 1} = x \; .$$

To obtain the right behavior of χ' at -1 and 1, we have to choose

$$c^{p-1} = \int_0^{+\infty} \frac{ds}{|s|^p + 1}\,.$$

Next a more careful study of χ' for t close to 1 shows that

$$\chi'(x) \sim K(c,p)(1-x)^{(1-p)^{-1}}$$

Hence χ' is integrable if $(1-p)^{-1} > -1$ i.e. for $p > 2$. This is (one of) the reason(s) why the condition $p > 2$ comes out.

Now choose any smooth solution u of the parabolic pde such that $u(x,0) \leq \chi(x)$ on $[-1,1]$: because $\chi'(\pm 1) = \pm\infty$, ± 1 cannot be a local maximum point of $u-\chi$ and it is easy to deduce

$$u(x,t) \leq \chi(x) + c^p t \quad \text{in } [-1,1] \times (0,+\infty)$$

In other words, $\chi(\cdot) + c^p t$ is the maximal subsolution (and solution) of the equation with initial data χ.

Main consequence : you cannot solve the Diririchlet problem with any boundary data...

The above example is a particular case in dimension 1 of results of Lasry & Lions : they prove that, for $p > 2$, in any dimension, the stationary problem admits a $C^2(\Omega)$ maximal solution which can be extended continuously to $\overline{\Omega}$.

We solve above a so-called "ergodic problem" for an equation with state-constraint boundary conditions (a terminology to be explained later on...)

Remark 3 : If we consider the (approximate ?) problem

$$\begin{aligned} u^\varepsilon_t - \Delta u^\varepsilon + \inf(|Du^\varepsilon|^p, \varepsilon^{-1}) &= 0 \quad \text{in } \Omega \times (0, +\infty) \\ u^\varepsilon(x,0) &= u_0(x) \quad \text{on } \overline{\Omega} \\ u^\varepsilon(x,t) &= \varphi(x,t) \quad \text{on } \partial\Omega \times (0,+\infty) \end{aligned}$$

then there exists a solution which is defined for all times and which satisfies

$$||u^\varepsilon||_\infty \leq \max(||u_0||_\infty, ||\varphi||_\infty)$$

but what kind of problem solve $u = \lim_\varepsilon \ u^\varepsilon$?

Remark 4 (and last) : The solution exists (for all time) because we have a formula !

We denote by $(X_t)_t$ the solution of the controlled stochastic differential equation

$$dX_s = \alpha_s dt + dW_s \text{ for } s > 0,\ X_0 = x \in \Omega$$

where $(\alpha_s)_s$, the control, is some progressively measurable process.

Then we define the value function of the exit time control problem by

$$U(x,t) = \inf_{(\alpha_s)_s} I\!E_x\Big[\int_0^\tau (p-1)p^{-\tilde{p}}|\alpha_s|^{\tilde{p}}ds + 1\!\!1_{\{\tau\le t\}}\varphi(X_\tau, t-\tau) + 1\!\!1_{\{\tau>t\}}u_0(X_t)\Big]$$

where τ is the first exit time of the trajectory $(X_s)_s$ from Ω, $\tilde{p} = \dfrac{p}{p-1}$ is the conjugate exponent to p.

Formally, U is the solution of our Dirichlet problem !

This formula leads to several questions, in particular if φ is large

(i) Is it possible to control the Brownian motion with a process $(\alpha_s)_s$?

(NB : The Brownian motion has infinite variations almost surely while the process $(\alpha_s)_s$ has bounded variations...)

(ii) If yes, is it worth doing it (in terms of cost) ?

• If p is "small", $\tilde{p}$ is large and the cost of the large controls α_s is large. So it is more interesting to avoid paying this large cost and to let the Brownian motion do whatever he wants, even if φ is large...

• On the contrary, if p is "large", $\tilde{p}$ is small and the cost of the large controls α_s is cheaper... So, if φ is large, it might be more interesting to control the Brownian motion to avoid paying φ.

This is why one cannot solve the Dirichlet problem for large φ : the "natural" solution IS NOT equal to φ on the boundary !

But what kind of problem U solve ? and is it the limit of the u^ε ?

The generalized Dirichlet boundary conditions (in the viscosity solutions sense)

Formally they can be written as

$$\min(u_t - \Delta u + |Du|^p, u - \varphi) \leq 0 \quad \text{on } \partial\Omega \times (0, +\infty),$$

and

$$\max(u_t - \Delta u + |Du|^p, u - \varphi) \geq 0 \quad \text{on } \partial\Omega \times (0, +\infty).$$

State-constraint boundary conditions are those obtained by taking, at least formally, $\varphi \equiv +\infty$, i.e.

$$u_t - \Delta u + |Du|^p \geq 0 \quad \text{on } \partial\Omega \times (0, +\infty).$$

(NB : in fact $\dfrac{\partial u}{\partial n} = +\infty\,!$)

Theorem : For any $u_0 \in C(\overline{\Omega})$ and $\varphi \in C(\partial\Omega \times [0, +\infty])$ satisfying the compatibility condition on $\partial\Omega \times \{0\}$, the value function U is continuous on $\overline{\Omega} \times [0, +\infty]$ and it is the unique solution of the generalized Dirichlet problem.

Moreover this problem has a "Strong Comparison Result" which implies that the solutions of any "reasonable" approximate problems converge to U (in particular the u^ε's).

Comments :

• **"Strong Comparison Result"** means that one can show that **any** upper-semi-continuous subsolution of the generalized Dirichlet problem is less than **any** lower-semi-continuous supersolution of the generalized Dirichlet problem

$\Longrightarrow$ a Maximum Principle for semi-continuous viscosity sub and supersolutions.

• **Useful ?** Yes because the **"Half-Relaxed Limits Method"** says that if $(v^\varepsilon)_\varepsilon$ are solutions of any "reasonable" approximate problems, then

$$\overline{v}(x,t) := \limsup_{\substack{(y,s)\to(x,t)\\ \varepsilon\to 0}} v^\varepsilon(y,s) \quad , \quad \underline{v}(x,t) := \liminf_{\substack{(y,s)\to(x,t)\\ \varepsilon\to 0}} v^\varepsilon(y,s)$$

are respectively sub and supersolution of the generalized Dirichlet problem.

Therefore the "Strong Comparison Result" yields

$$\overline{v}(x,t) \leq U(x,t) \leq \underline{v}(x,t) \quad \text{in } \Omega \times [0,+\infty)$$

(one has to be careful with the boundary layer...) and because of the definition of $\overline{v}$ and $\underline{v}$, this implies the local uniform convergence of the v^ε to U in Ω.

NB : In fact the "Strong Comparison Result" is the cornerstone of everything...

“Strong Comparison Results” for the Generalized Dirichlet problem

• Far easier to prove for Neumann boundary conditions : (very) general SCR exists for linear and nonlinear Neumann boundary conditions without particular conditions on the equations (P.L. Lions, H. Ishii and GB).

• The difficulty comes from the fact that the sub AND supersolutions are discontinuous : if one is continuous, it is far easier (E. Rouy, P.E Souganidis and GB)

• The first proof given by Soner was using a cone condition on the boundary, i.e. for the subsolution u : there exists a constant C such that, for any $(x, t) \in \partial\Omega \times (0, +\infty)$

$$u(x, t) = \lim_k u(x_k, t_k) \quad \text{with } |x - x_k| \leq C d(x_k, \partial\Omega)$$

This property (which is needed only for the subsolution in our case but which may be needed also for the supersolution in other cases) is the KEY RESULT of our work (and would hold in more general cases...).

- **Other SCR for the Dirichlet problem**

– **First-Order Equations : H. Ishii, B. Perthame & GB, H. Ishii & S. Koike.**

– **Semi-linear case : J. Burdeau & GB**

– **Hamilton-Jacobi-Bellman : M. Katsoulakis, E. Rouy & GB**

- **Open problem (or almost open) : the case of quasilinear equations**

$$-\mathrm{Tr}(A(Du)D^2u) + \cdots$$

despite several results of Da Lio, Da Lio & GB.