Entropy and Zeta Functions for One-dimensional Multi-layer Cellular Neural Networks

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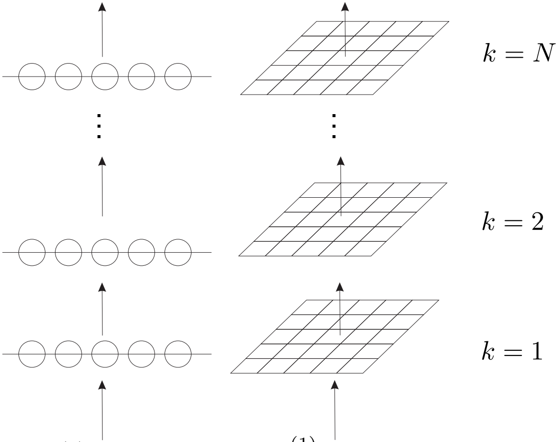
[*] Joint work with Prof. Song-Sun Lin, Prof. Yin-Heng Lin and Mr. Chih-Hung Chang

Outline

Outline

- L.O. Chua and L. Yang



$\{u_\ell^{(1)}\}$: input
1-d MCNN

$\{u_{ij}^{(1)}\}$: input
2-d MCNN

One-dimensional Multi-layer CNN

(1)
$$\begin{cases} \dfrac{dx_i^{(1)}}{dt} = -x_i^{(1)} + \sum\limits_{|k|\leq d} a_k^{(1)} y_{i+k}^{(1)} + \sum\limits_{|k|\leq d} b_k^{(1)} u_{i+k}^{(1)} + z^{(1)}, \\ \dfrac{dx_i^{(2)}}{dt} = -x_i^{(2)} + \sum\limits_{|k|\leq d} a_k^{(2)} y_{i+k}^{(2)} + \sum\limits_{|k|\leq d} b_k^{(2)} u_{i+k}^{(2)} + z^{(2)}, \\ \qquad\vdots \\ \dfrac{dx_i^{(N)}}{dt} = -x_i^{(N)} + \sum\limits_{|k|\leq d} a_k^{(N)} y_{i+k}^{(N)} + \sum\limits_{|k|\leq d} b_k^{(N)} u_{i+k}^{(N)} + z^{(N)}, \end{cases}$$

where

$$y_l^{(k)} = u_l^{(k+1)}, \quad 1 \leq k \leq N-1$$

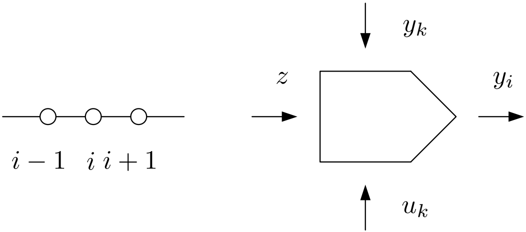
and

$$\{u_l^{(1)}\} : \text{ input data.}$$

Consider $N = 1$ and $d = 1$,

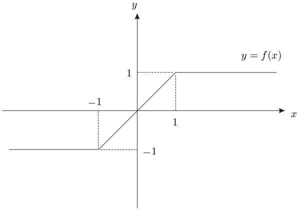
$$\frac{dx_i}{dt} = -x_i + a_l y_{i-1} + a y_i + a_r y_{i+1} + b_l u_{i-1} + b u_i + b_r u_{i+1} + z.$$

The diagram of this CNN model:



- Output function

$$y = f(x) = \frac{1}{2}(|x+1| - |x-1|).$$



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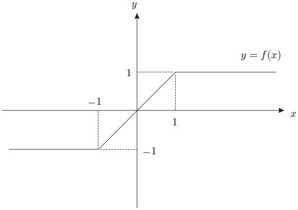

- Initial conditions

$$u_i^{(k)}(0) = u_i^{(k)} \text{ and } x_i^{(k)}(0) : \text{ constant}$$

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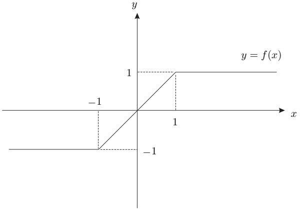

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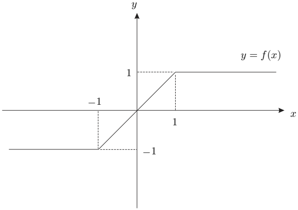

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  - $A^{(k)}$: feedback template.

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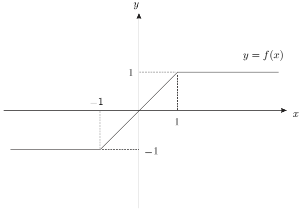

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$$u_i^{(k)}(0) = u_i^{(k)} \text{ and } x_i^{(k)}(0) : \text{ constant}$$

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  - $A^{(k)}$: feedback template.
  - $B^{(k)}$: controlling template.

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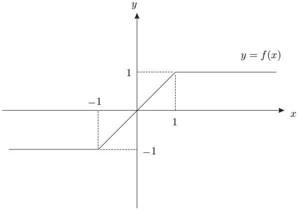

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  - $A^{(k)}$: feedback template.
  - $B^{(k)}$: controlling template.
  - $z^{(k)}$: threshold.

Definition 1.1

- Mosaic patterns (Output patterns)

$$\{(y_i^{(k)})\}_{\substack{i\in\mathbb{Z}\\ 1\le k\le N}} \Longleftrightarrow |y_i^{(k)}| = 1$$

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Consider $A^{(k)} = (a_l^{(k)}, a^{(k)}, a_r^{(k)})$, $B^{(k)} = (b_l^{(k)}, b^{(k)}, b_r^{(k)})$

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Parameters space $\{(A^{(1)}, B^{(1)}, z^{(1)}),\ \cdots, (A^{(N)}, B^{(N)}, z^{(N)})\}$

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In (1), consider stationary solutions:

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In (1), consider stationary solutions:

MCNN $\Longrightarrow$ N-coupled map lattice

(2) $$\begin{cases} x_i^{(1)}=a_l^{(1)}y_{i-1}^{(1)}+a^{(1)}y_i^{(1)}+a_r^{(1)}y_{i+1}^{(1)} \\ \qquad +b_l^{(1)}u_{i-1}^{(1)}+b^{(1)}u_i^{(1)}+b_r^{(1)}u_{i+1}^{(1)}+z^{(1)}, \\ x_i^{(2)}=a_l^{(2)}y_{i-1}^{(2)}+a^{(2)}y_i^{(2)}+a_r^{(2)}y_{i+1}^{(2)} \\ \qquad +b_l^{(2)}y_{i-1}^{(1)}+b^{(2)}y_i^{(1)}+b_r^{(2)}y_{i+1}^{(1)}+z^{(2)}, \\ \qquad \vdots \\ x_i^{(N)}=a_l^{(N)}y_{i-1}^{(N)}+a^{(N)}y_i^{(N)}+a_r^{(N)}y_{i+1}^{(N)} \\ \qquad +b_l^{(N)}y_{i-1}^{(N-1)}+b^{(N)}y_i^{(N-1)}+b_r^{(N)}y_{i+1}^{(N-1)}+z^{(N)}. \end{cases}$$

Question

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(II) Complexity of $\{$ outputs $\} \equiv \mathcal{Y}$.
(i) Spatial entropy.
(ii) Zeta function.

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One-layer CNN

In (1), consider $N = 1$ and $d = 1$.

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Consider the stationary solutions,

$$0 = -x_i + a_l y_{i-1} + a y_i + a_r y_{i+1} + b_l u_{i-1} + b u_i + b_r u_{i+1} + z. \tag{4}$$

Output patterns

- Mosaic patterns with $B = 0$:

$$\begin{aligned} y_i = 1 &\quad \Leftrightarrow \quad (a-1) + z + (a_l y_{i-1} + a_r y_{i+1}) > 0. \\ y_i = -1 &\quad \Leftrightarrow \quad (a-1) - z - (a_l y_{i-1} + a_r y_{i+1}) > 0. \end{aligned}$$

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- Mosaic patterns with $B \neq 0$:

$$\begin{aligned} y_i = 1 \quad\Leftrightarrow\quad & (a-1) + z > \\ & -(a_l y_{i-1} + a_r y_{i+1} + b_l u_{i-1} + b u_i + b_r u_{i+1}). \end{aligned}$$

$$\begin{aligned} y_i = -1 \quad\Leftrightarrow\quad & (a-1) - z > \\ & (a_l y_{i-1} + a_r y_{i+1} + b_l u_{i-1} + b u_i + b_r u_{i+1}). \end{aligned}$$

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One-layer CNN without input $(B = 0)$

Procedures

1. Partition of parameters space.

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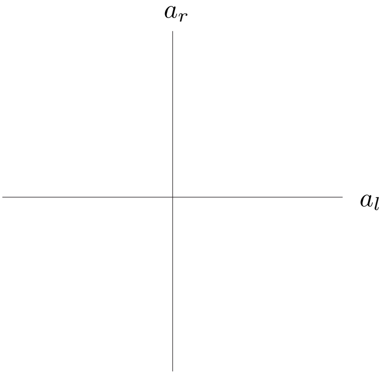
6. Zeta function.

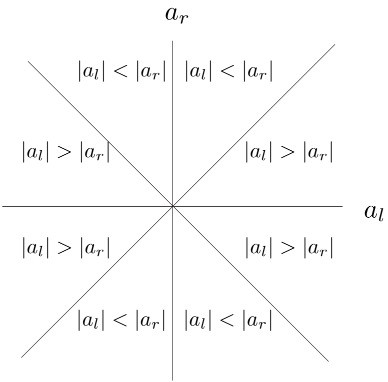
(I) Partition of parameters space (a_l, a_r, a, z)

1. Sign of parameters:



(I) Partition of parameters space (a_l, a_r, a, z)

2. Length of parameters:



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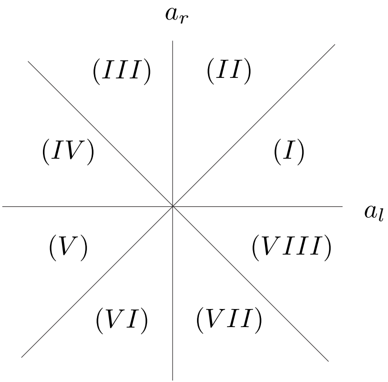
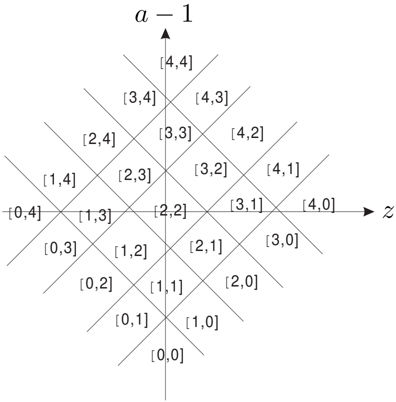


(I) Partition of parameters space (a_l, a_r, a, z)

4. Relations between a, z and other parameters $[m,n]_{(J)}$:



(II) Ordering matrix of local patterns

$$\mathbb{X}_{3\times 1} = \begin{array}{c} \\ \boxed{0\ 0} \\ \boxed{0\ 1} \\ \boxed{1\ 0} \\ \boxed{1\ 1} \end{array} \begin{array}{cccc} \boxed{0\ 0} & \boxed{0\ 1} & \boxed{1\ 0} & \boxed{1\ 1} \\ \boxed{0\ 0\ 0} & \boxed{0\ 0\ 1} & \varnothing & \varnothing \\ \varnothing & \varnothing & \boxed{0\ 1\ 0} & \boxed{0\ 1\ 1} \\ \boxed{1\ 0\ 0} & \boxed{1\ 0\ 1} & \varnothing & \varnothing \\ \varnothing & \varnothing & \boxed{1\ 1\ 0} & \boxed{1\ 1\ 1} \end{array}$$

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The transition matrix

$$\mathbb{T} \equiv \mathbb{T}(A, z) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}. \quad (5)$$

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$\Longrightarrow$ subshift of finite type of $\{0, 1\}^{\mathbb{Z}^1}$.

(IV) Subshift of finite type

The same region as last example,
$G = (\mathcal{V}(G), \mathcal{E}(G))$, where

$$\begin{aligned} \mathcal{V}(G) &= \{v_1, v_2, v_3, v_4\} = \{00, 01, 10, 11\}, \\ \mathcal{E}(G) &= \{e_{ij} \,|\, 1 \le i, j \le 4, t_{ij} = 1\}. \end{aligned}$$

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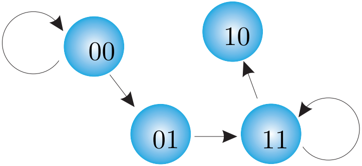

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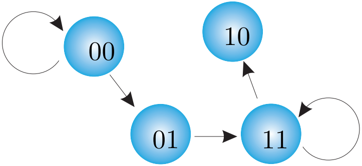

$\mathbb{T} = \mathbb{T}(A, z)$: transition matrix of $G = G(A, z)$.

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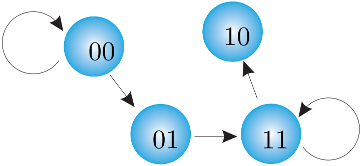

$\mathbb{T} = \mathbb{T}(A, z)$: transition matrix of $G = G(A, z)$.

$$\begin{aligned} \Sigma_n(A, z) &= \{\text{patterns of length } n\} \\ &= \{e_{i_1 i_2} e_{i_2 i_3} \cdots e_{i_{n-2} i_{n-1}} |\, t_{i_k i_{k+1}} = 1, 1 \le k \le n-2\}. \end{aligned}$$

(V) Spatial entropy

Given $(A, z) \Rightarrow \mathcal{B}(A, z) \Rightarrow \mathbb{T}$

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$$\Rightarrow \Gamma_{n\times 1} = |\mathbb{T}^{n-2}|, \quad \forall n \geq 3$$

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$$\begin{aligned} h((A,z)) &\equiv \lim_{n\to\infty} \frac{\log \Gamma_{n\times 1}}{n} \\ &= \lim_{n\to\infty} \frac{\log |\mathbb{T}^{n-2}|}{n} = \log \rho(\mathbb{T}), \end{aligned}$$

where $\rho(\mathbb{T})$ is the maximum eigenvalue of $\mathbb{T}$.

(V) Spatial entropy

Given $(A, z) \Rightarrow \mathcal{B}(A, z) \Rightarrow \mathbb{T}$

$\Gamma_{n\times 1}$: number of admissible patterns on $\mathbb{Z}_{n\times 1}$

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where $\rho(\mathbb{T})$ is the maximum eigenvalue of $\mathbb{T}$.

$$h(\mathbb{T}) \begin{cases} > 0 : & \text{spatial chaos} \\ = 0 : & \text{pattern formation} \end{cases}$$

(VI) Zeta function

Definition 2.1

The zeta function of $\mathbb{T}$ is defined by

$$\zeta(t) := \exp\left(\sum_{n=1}^{\infty} \frac{p_n}{n} t^n\right), \tag{6}$$

where p_n is the number of periodic patterns with period n in $\mathbb{Z}^1$.

Lemma 2.2

$$p_n = tr(\mathbb{T}^n). \tag{7}$$

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Theorem 2.3

$$\begin{aligned} \zeta(t) &= \exp(\sum_{n=1}^{\infty} \frac{tr(\mathbb{T}^n)}{n} t^n) \\ &= (\det(I - t\mathbb{T}))^{-1}. \end{aligned} \tag{8}$$

Outline

One-layer CNN with input ($B \neq 0$)

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One-layer CNN with input ($B \neq 0$)

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4. Labeled graph $\Longrightarrow$ Sofic shift.

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1. Sign of parameters (a_l, a_r, b_l, b, b_r).

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1. Sign of parameters (a_l, a_r, b_l, b, b_r).
2. Length of $(|a_l|, |a_r|, |b_l|, |b|, |b_r|)$.
3. Relations between $(|a_l|, |a_r|, |b_l|, |b|, |b_r|)$.

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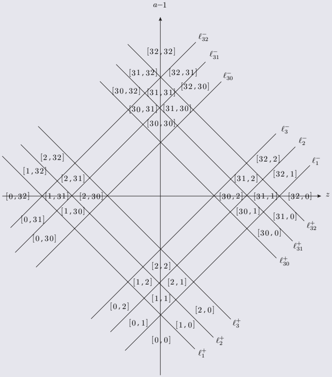
3. Relations between $(|a_l|, |a_r|, |b_l|, |b|, |b_r|)$.
4. Relations between a, z and other parameters $[m, n]_{(J)}$, $J = [j_1, j_2, j_3]$.



Theorem 2.4

For $(A,B,z)\in\mathbb{R}^7$*, there is a unique region* $[m,n]_{(J)}$ *defines the basic set of admissible local patterns* $\mathcal{B}(A,B,z)$*, where* $0\leq m,n\leq 32$, $J=[j_1,j_2,j_3]$, $1\leq j_1\leq 32$, $1\leq j_2\leq 120$, $1\leq j_3\leq 231$.

Example 2.5

Pick $A = [a_l, a, a_r]$ and $B = [b_l, b, b_r]$ satisfying

(1) $a_l > b_l > a_r > b > b_r > 0;$

(2) $a_l + b_r < b_l + a_r$, $a_l + b > a_r + b_l + b_r;$

(3) $a_r + b + b_r < a_l < b_l + b;$

(4) $b_l + b_r < a_r + b$, $b_l > a_r + b_r$, $a_r > b + b_r$

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$$
\begin{array}{cccc}
\begin{array}{ccc} - & \ominus & - \\ - & \boxplus & - \end{array} &
\begin{array}{ccc} - & \ominus & - \\ - & \boxplus & + \end{array} &
\begin{array}{ccc} - & \ominus & - \\ + & \boxminus & + \end{array} &
\begin{array}{ccc} - & \ominus & + \\ - & \boxplus & - \end{array} \\[1em]
\begin{array}{ccc} - & \ominus & + \\ - & \boxplus & + \end{array} &
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\begin{array}{ccc} + & \oplus & + \\ + & \boxminus & + \end{array} &
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\begin{array}{ccc} - & \oplus & + \\ - & \boxplus & + \end{array}
\end{array}
$$

(II) Ordering matrix of $\mathbb{X}_{3\times 2}$

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- Given $\mathcal{B} \equiv \mathcal{B}\left((A,B,z)\right) \subset \{0,1\}^{\mathbb{Z}_{3\times 2}}$, hope to generate $\Sigma\left(\mathcal{B}\right)$.

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where $(y \circ u)$ denote the $\infty \times 2$ admissible patterns generated from $\mathcal{B}$, i.e., $\begin{array}{c} \cdots y_{-1} y_0 y_1 \cdots \\ \cdots u_{-1} u_0 u_1 \cdots \end{array} \in \Sigma\left(\mathcal{B}\right)$.

- The ordering matrix $\mathbb{X}_{3\times 2} = \mathbb{X}\left(\mathcal{B}\right) =$

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 & \begin{array}{cccc}
\boxed{\substack{00\\00}}\ \boxed{\substack{00\\01}}\ \boxed{\substack{00\\10}}\ \boxed{\substack{00\\11}} &
\boxed{\substack{01\\00}}\ \boxed{\substack{01\\01}}\ \boxed{\substack{01\\10}}\ \boxed{\substack{01\\11}} &
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\boxed{\substack{11\\00}}\ \boxed{\substack{11\\01}}\ \boxed{\substack{11\\10}}\ \boxed{\substack{11\\11}}
\end{array} \\
\begin{array}{c}
\begin{matrix}\boxed{\substack{00\\00}}\\ \boxed{\substack{00\\01}}\\ \boxed{\substack{00\\10}}\\ \boxed{\substack{00\\11}}\end{matrix} \\
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\begin{matrix}\boxed{\substack{10\\00}}\\ \boxed{\substack{10\\01}}\\ \boxed{\substack{10\\10}}\\ \boxed{\substack{10\\11}}\end{matrix} \\
\begin{matrix}\boxed{\substack{11\\00}}\\ \boxed{\substack{11\\01}}\\ \boxed{\substack{11\\10}}\\ \boxed{\substack{11\\11}}\end{matrix}
\end{array} &
\left(\begin{array}{c:c:c:c}
\begin{pmatrix}
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\boxed{\substack{010\\100}} & \boxed{\substack{010\\101}} & \varnothing & \varnothing \\
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\boxed{\substack{011\\100}} & \boxed{\substack{011\\101}} & \varnothing & \varnothing \\
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\end{pmatrix} \\
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\boxed{\substack{100\\000}} & \boxed{\substack{100\\001}} & \varnothing & \varnothing \\
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\end{pmatrix} &
\begin{pmatrix}
\boxed{\substack{101\\000}} & \boxed{\substack{101\\001}} & \varnothing & \varnothing \\
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\end{pmatrix} &
\varnothing & \varnothing \\
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\end{pmatrix}
\end{array}\right)
\end{array}$$

Remark

One can see that there is self-similarity on the ordering matrix of CNN with input.

(III) Labeled graph

Theorem 2.6

One-layer CNN with input $\mathcal{U}$ is a sofic shift.

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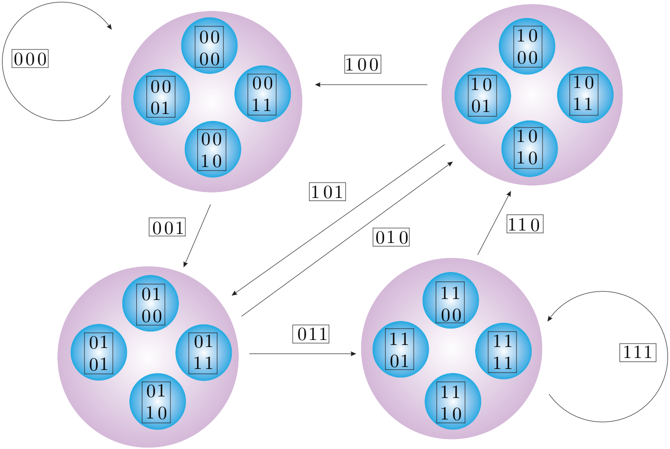

(IV) Transition matrix

Example 2.7 (Continued)

$\mathbb{T}(\mathcal{B})$ can be constructed w.r.t $\mathbb{X}(\mathcal{B})$ as follows:

$$\mathbb{T} = \begin{bmatrix} T_1 & T_1 & 0 & 0 \\ 0 & 0 & 0 & T_3 \\ T_2 & 0 & 0 & 0 \\ 0 & 0 & T_1 & T_1 \end{bmatrix},$$

where $T_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $T_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and

$$T_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Outline

Why Sofic?

Given A, B and z, we have the basic set $(B) \equiv \mathcal{B}(A,B,z)$

1. $|\mathbb{T}^{n-2}(\mathcal{B})|$ =the number of all admissible patterns in $\Sigma_{n\times 2}(\mathcal{B})$

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3. Once

$$\mathbb{T} = \begin{bmatrix} T_1 & T_2 & 0 & 0 \\ 0 & 0 & T_3 & T_4 \\ T_5 & T_6 & 0 & 0 \\ 0 & 0 & T_7 & T_8 \end{bmatrix}$$

is constructed, let $\mathcal{S} = \{s_{000}, s_{001}, s_{010}, s_{011}, s_{100}, s_{101}, s_{110}, s_{111}\}$, then symbolic transition matrix $\mathbb{S}$ is:

$$\mathbb{S} = \begin{bmatrix} s_{000}T_1 & s_{001}T_2 & 0 & 0 \\ 0 & 0 & s_{010}T_3 & s_{011}T_4 \\ s_{100}T_5 & s_{101}T_6 & 0 & 0 \\ 0 & 0 & s_{110}T_7 & s_{111}T_8 \end{bmatrix}.$$

Theorem 2.8

Y_U is conjugate to $X_{\mathbb{S}}$ under $\phi : Y_U \to X_{\mathbb{S}}$, where ϕ is a factor map induced by the sliding block code $\Phi\left(y_0 y_1 y_2\right) = s_{y_0 y_1 y_2}$.

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Theorem 2.9

If X and Y are shift spaces, and X is conjugate to Y under conjugacy ϕ, then

- $h\left(X\right) = h\left(X\right);$
- $\zeta_{\sigma_X}\left(X\right) = \zeta_{\sigma_Y}\left(Y\right)$

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Theorem 2.9

If X and Y are shift spaces, and X is conjugate to Y under conjugacy ϕ, then

- $h(X) = h(X)$;
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- Thus the problem of computation the **entropy** and **zeta function** of Y_U is equivalent to compute the these two invariants of $X_{\mathbb{S}}$.

Outline

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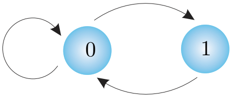
Introduction to Sofic Shift

Definition 3.1

A labeled graph $\mathcal{G}$ is a pair $(G, \mathcal{L})$, where G is a graph with edge $\mathcal{E}$, and the labeling $\mathcal{L} : \mathcal{E} \to \mathcal{A}$ assigns to each edge e of G a label $\mathcal{L}(e)$ from the finite alphabet $\mathcal{A}$. The underlying graph of $\mathcal{G}$ is G.



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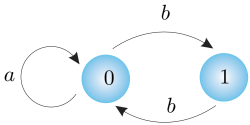
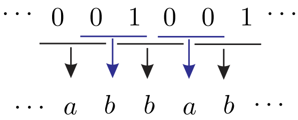


- If $\zeta = \cdots e_{-1}e_0e_1\cdots$ is a bi-infinite word in G, define the label of the walk ζ to be

$$\mathcal{L}_\infty = \cdots \mathcal{L}\left(e_{-1}\right)\mathcal{L}\left(e_0\right)\mathcal{L}\left(e_1\right)\cdots \in \mathcal{A}^{\mathbb{Z}}.$$



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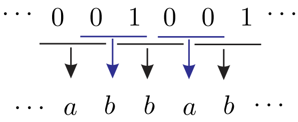

- Thus, $X_{\mathcal{G}}$ is defined,

$$X_{\mathcal{G}} = \left\{x \in \mathcal{A}^{\mathbb{Z}} : x = \mathcal{L}_\infty\left(\zeta\right) \text{ for some } \zeta \in X_G\right\}.$$

Outline

1. Introduction
2. One-layer CNN
   - One-layer CNN
   - Without input
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   - Why sofic?
3. Sofic shift
   - Introduction to sofic shift
   - Entropy for shift space
   - Zeta function
   - Application to one-layer CNN with input
   - Difference between CNN with and without input
4. Multi-layer CNN
   - Two-layer CNN with input
   - Multi-layer CNN
5. New phenomena
6. Recent works and future plans
7. Reference

Entropy for Shift Space

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- The entropy of X is the growth rate of n-block in X, i.e., it measures the complexity of X.
- For $N-$layer CNN, in some region of parameter space, we compute the entropy of the output space $h(Y_U)$, where $Y_U = \left\{ (y_i)_{i\in\mathbb{Z}} \mid \exists (u_i)_{i\in\mathbb{Z}} \text{ s.t. } (y \circ u) \in \Sigma(\mathcal{B}) \right\}$

Computation of Entropy for SFT

Theorem 3.2

If X is a SFT, and $\mathbb{T}$ is its transition matrix, then

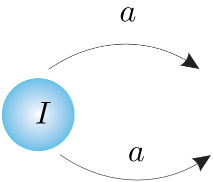
$$h\left(X\right)=\log\rho\left(\mathbb{T}\right).$$

where $\rho\left(\mathbb{T}\right)$ is the maximal eigenvalue of $\mathbb{T}$.

Computation of Entropy for Sofic Shift

Definition 3.3

A label set $\mathcal{G} = (G, \mathcal{L})$ is right resolving if for every vertex I of G, the edge from I carry different labels.



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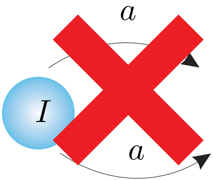

Theorem 3.4

Given Y_U, *and* $\mathbb{S} = (G_{\mathbb{T}}, \mathcal{L})$ *is the sofic shift induced by* Y_U, *if* $\mathbb{S}$ *is* ***right-resolving****, then the entropy of* $X_{\mathbb{S}}$ *is*

$$h(X_{\mathbb{S}}) = \log \rho(\mathbb{T}).$$

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where $\rho\left(\mathbb{T}\right)$ is the maximal eigenvalue of $\mathbb{T}$.

- In this case, $h\left(Y_U\right) = h\left(X_{\mathbb{S}}\right) = \log \rho\left(\mathbb{T}\right).$

Theorem 3.5

Given $\mathbb{S} = (G_{\mathbb{T}}, \mathcal{L})$ is a labeled graph which is not right resolving, then there is a sofic shift $\mathcal{H} = (G_{\mathbb{H}}, \mathcal{L})$ which is right resolving that is conjugate to $X_{\mathbb{S}}$, i.e., $X_{\mathbb{S}} = X_{\mathcal{H}}$.

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- In this case $h(Y_U) = h(X_{\mathbb{S}}) = h(X_{\mathcal{H}}) = \log \rho(\mathbb{H})$.

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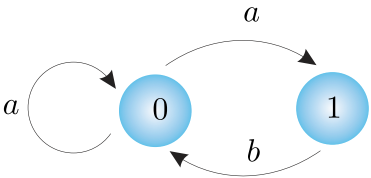
- In this case $h(Y_U) = h(X_{\mathbb{S}}) = h(X_{\mathcal{H}}) = \log \rho(\mathbb{H})$.
- The method for constructing $\mathcal{H} = (G_{\mathbb{H}}, \mathcal{L})$ is called “Subset Construction”.

Subset Construction

Example 3.6

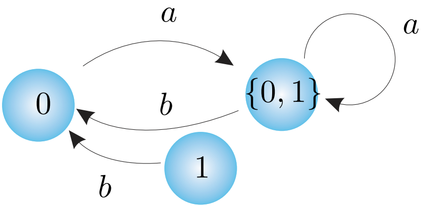
For a sofic shift with the sign matrix is

$$B = \begin{bmatrix} a & a \\ b & \varnothing \end{bmatrix}, \; A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$



Introduce a new symbol $\{0,1\}$,and a new symbolic adjacency matrix C indexed by $\{0,1,\{0,1\}\}$as

$$C = \begin{bmatrix} a & b & \varnothing \\ a & \varnothing & \varnothing \\ \varnothing & b & \varnothing \end{bmatrix}$$



Thus in this case C is right resolving, and $X_{\mathcal{G}_B} \simeq X_{\mathcal{G}_C}$.

- Such construction is not unique, all of them are conjugate.

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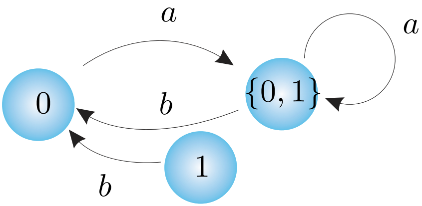

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- Such construction is not unique, all of them are conjugate.
- Minimal subset construction

For computation the entropy of $\mathcal{G}_B = (G_A, \mathcal{L})$, define the transition matrix of C by

$$D = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

we have

$$h\left(X_{\mathcal{G}_B}\right) = h\left(X_{\mathcal{G}_C}\right) = \log \rho\left(D\right) = \log \frac{1+\sqrt{5}}{2}.$$

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4. Multi-layer CNN
   - Two-layer CNN with input
   - Multi-layer CNN
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7. Reference

Zeta Functions for SFT

Definition 3.7

Let (M, ϕ) be a dynamical system for which $p_n(\phi) < \infty$ for all $n \geq 1$. The zeta function $\zeta_\phi(t)$ is defined as

$$\zeta_\phi(t) = \exp\left(\sum_{n=1}^{\infty} \frac{p_n(\phi)}{n} t^n\right).$$

Theorem 3.8

Let A be an $r \times r$ nonnegative integer matrix, $\chi_A(t)$ is characteristic polynomial, then

$$\zeta_{\sigma_A}(t) = \frac{1}{t^r \chi_A(t^{-1})} = \frac{1}{\det(Id - tA)}$$

Thus the zeta function of a SFT is the reciprocal of a polynomial.

Notations:

$\mathcal{G}=(G,\mathcal{L})$: right resolving labeled graph
r : the number of vertices in G
$\mathcal{V}=\{1,\cdots,r\}$
B : the symbolic transition matrix of $X_{\mathcal{G}}$
A : obtained from B by letting all the symbol equal to 1

1. $F=\{f_1\cdots f_j\}$
   π : permutation act on F
   i.e., $\pi\left(F\right)=\left(f_{i_1},\cdots,f_{i_j}\right)$ with $\pi\left(f_k\right)=f_{i_k}$ for $k=1,\cdots,j$.

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   π : permutation act on F
   i.e., $\pi(F)=\left(f_{i_1},\cdots,f_{i_j}\right)$ with $\pi(f_k)=f_{i_k}$ for $k=1,\cdots,j$.
2. Define the

$$sgn(\pi)=\begin{cases}1 & \text{if } \pi \text{ is even}\\ -1 & \text{if } \pi \text{ is odd}\end{cases}$$

Signed subset matrices:

1. Construct $\mathcal{G}_j$ with $\{\pm a : a \in \mathcal{A}\}$, where $1 \leq j \leq r = |\mathcal{V}|$.

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2. $|\mathcal{V}_j| = \binom{r}{j}$: the number of vertices of $\mathcal{G}_j$.
3. If there is a edge a sent $\mathcal{I} = \{I_1, \cdots, I_j\}$ to $\mathcal{J} = \{J_1, \cdots, J_j\}$, then there is a labeling a from $\mathcal{I}$ to $\mathcal{J}$ in $\mathcal{G}_j$ if $(J_1, \cdots, J_j)$ is even, otherwise, label it by $-a$.

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4. We get $\mathcal{G}_j, B_j$ and A_j.

Theorem 3.9

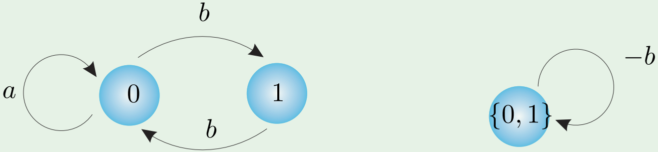
Let $\mathcal{G}$ be a right resolving labeled graph with r vertices, and let A_j be its jth sign subset matrix. Then

$$\zeta_{\sigma_{\mathcal{G}}}(t) = \prod_{j=1}^{r} \left[\det\left(Id - tA_j\right)\right]^{(-1)^j}$$

- Thus the zeta function for a sofic shift is a *rational function*.

Example 3.10 (Even Shift)

If $\mathcal{G} = (G_A, \mathcal{L})$, with $B = \begin{bmatrix} a & b \\ b & \varnothing \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, then $\mathcal{G}_1 = \mathcal{G}$. Moreover, $\mathcal{G}_2$ can be constructed with $B_2 = [-b]$ and $A_2 = [-1]$.



Thus, the zeta function can be computed as

$$\zeta_{\sigma_{\mathcal{G}}}(t) = \frac{1+t}{1-t-t^2}.$$

Outline

Application to one-layer CNN with input

Theorem 3.11

Given A, B, z, *and* $\mathcal{U}$, *let* $\widetilde{\mathbb{T}} = \mathbb{T}((A, B, z))$, $\mathbb{T} = \mathbb{T}((A, B, z); \mathcal{U}) \in \mathcal{M}_{16}(\mathbb{R})$ *and* $\mathbb{U} = \mathbb{T}(\mathcal{U}) \in \mathcal{M}_4(\mathbb{R})$ *be the transition matrices, then*

$$\mathbb{T} = \widetilde{\mathbb{T}} \circ (E_4 \otimes \mathbb{U}), \tag{9}$$

where $\circ$ *and* $\otimes$ *are the Hadamard product and Kronecker product, resp.,* $E_4 = (e_{ij}) \in \mathcal{M}_4(\mathbb{R})$ *with* $e_{ij} = 1$ *for all* i, j.

Computation of entropy for CNN with input

Example 3.12 (Continued)

Same example as before, the transition matrix is

$$\mathbb{T} = \begin{bmatrix} T_1 & T_1 & 0 & 0 \\ 0 & 0 & 0 & T_3 \\ T_2 & 0 & 0 & 0 \\ 0 & 0 & T_1 & T_1 \end{bmatrix},$$

where $T_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $T_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and

$$T_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

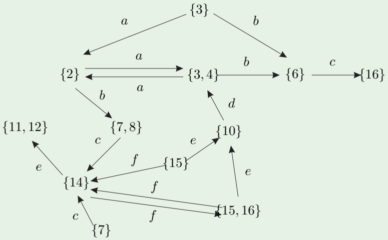
Computation of entropy for CNN with input

Example 3.12 (Continued)

Therefore, we have

$$\mathbb{S} = \begin{bmatrix} s_{000}T_1 & s_{001}T_1 & 0 & 0 \\ 0 & 0 & 0 & s_{011}T_3 \\ s_{100}T_2 & 0 & 0 & 0 \\ 0 & 0 & s_{110}T_1 & s_{111}T_1 \end{bmatrix}.$$

This sofic is not right resolving. Use subset construction:



Computation of entropy for CNN with input

Example 3.12 (Continued)

New right resolving sofic shift can be obtained as

$$\mathbb{H} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

then $h\left((A, B, z)\right) = \log \lambda > 0$
where λ is the maximal eigenvalue of $t^6 - 2t^4 + t^2 - 1 = 0$.

Computation of entropy for CNN with input

Example 3.12 (Continued)

For the computation of zeta function, we should construct the $k-$th sign subset matrices A_k of $\mathbb{H}$.

$$A_1 = \mathbb{H},\ A_2 = \begin{bmatrix} 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and $A_3 = \cdots = A_{12} = 0$, then

$$\zeta_{\sigma_{\mathbb{H}}} = \frac{(t+1)^2}{1-2t^2+t^4-t^6}.$$

Outline

Relations

Theorem 3.13

Given (A, B, z) , *then*

1. *If* $B = 0$, *then* $\mathcal{B}(A, B, z)$ *and transition matrix* $\mathbb{T}$ *can be uniquely defined and*

$$\Sigma(\mathcal{B}(A, B, z)) = X_{\mathbb{T}}.$$

Relations

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1. *If* $B = 0$*, then* $\mathcal{B}(A, B, z)$ *and transition matrix* $\mathbb{T}$ *can be uniquely defined and*

$$\Sigma(\mathcal{B}(A, B, z)) = X_{\mathbb{T}}.$$

2. *If* $B \neq 0$*,* $\mathcal{B}(A, B, z)$ *and transition matrix* $\mathbb{T}$ *can be uniquely defined. Let* $\mathcal{S} = \{s_i\}_{i \in I}$ *and construct* $\mathbb{S}$*, then*

$$Y_U = X_{\mathbb{S}},$$

where

$$Y_U = \{(y_i)_{i\in\mathbb{Z}} \mid \text{there exists } (u_i)_{i\in\mathbb{Z}} \text{ s.t. } (y \circ u) \in \Sigma(\mathcal{B})\}.$$

and $\mathbb{S} = (G_{\mathbb{T}}, \mathcal{S})$ *is the sofic shift with underling graph* $G_{\mathbb{T}}$ *and labeling by* $\mathcal{S}$*.*

- CNN without input $\Leftrightarrow$ SFT

- CNN without input $\Leftrightarrow$ SFT
  - Entropy

- CNN without input $\Leftrightarrow$ SFT
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Outline

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   - One-layer CNN
   - Without input
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   - Why sofic?
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7. Reference

Outline

MCNN: N-layer CNN with input

Equations

$$\frac{dx_i^{(n)}}{dt} = -x_i^{(n)} + \sum_{|k|\leq d} a_k y_{i+k}^{(n)} + \sum_{|k|\leq d} b_k u_{i+k}^{(n)} + z^{(n)}, \tag{10}$$

where $u_i^{(n)} = y_i^{(n-1)}$ for $1 \leq n \leq N$, and $u_i^{(0)} = u_i$ for $i \in \mathbb{Z}$.

Questions

For N-layer CNN with input:

- **Relation between i-th and $(\mathrm{i}+1)-$th layer?**

MCNN: N-layer CNN with input

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Questions

For N-layer CNN with input:

- **Relation between i-th and $(\mathrm{i}+1)-$th layer?**
- **Entropy?**
- **Zeta function?**

Partitions of the parameters spaces

Denote the parameters spaces and admissible local patterns of each layer by

$$\mathcal{P}^{(i)} = \{(A^{(i)}, B^{(i)}, z^{(i)})\},\ \mathcal{B}^{(i)}(A^{(i)}, B^{(i)}, z^{(i)}), \tag{11}$$

for $1 \leq i \leq N$. Let

$$\begin{aligned} A &= (A^{(1)}, A^{(2)}, \cdots, A^{(N)}), \\ B &= (B^{(1)}, B^{(2)}, \cdots, B^{(N)}), \\ \mathcal{P} &= (\mathcal{P}^{(1)}, \mathcal{P}^{(2)}, \cdots, \mathcal{P}^{(N)}). \end{aligned} \tag{12}$$

Partitions of the parameters spaces

Theorem 4.1

There exists $K \in \mathbb{N}$ and unique set of open subregions $\{P_k\}_{k=1}^{K}$ of $\mathcal{P}$ such that

(i) $\mathcal{P} = \bigcup_{k=1}^{K} \bar{P}_k$.

(ii) $P_k \bigcap P_j = \varnothing$ *for* $k \neq j$.

(iii) $\mathcal{B}(A, B, z) = \mathcal{B}(A', B', z') \Leftrightarrow (A, B, z), (A', B', z') \in P_k$ *for some* k.

- Linear Separation Theorem.
- Thus, $\mathcal{B}\left(A^{(i)}, B^{(i)}, z^{(i)}\right)$ for N–layer MCNN with input can be uniquely specified.

Transition matrix for $N = 2$

Given $(A^{(1)}, B^{(1)}, z^{(1)})$, $(A^{(2)}, B^{(2)}, z^{(2)})$ and $\mathcal{U}$, we obtain the admissible local patterns $\mathcal{B}^{(1)}$, $\mathcal{B}^{(2)}$ and the transition matrices $\mathbb{T}_1$, $\mathbb{T}_2$, $\mathbb{U}$.
Denote the transition matrix of $\mathcal{B}((A, B, z); \mathcal{U}) = \mathcal{B}^{(2)} * \mathcal{B}^{(1)}$ by $\mathbb{T}$.

Theorem 4.2

$$\mathbb{T} = (\mathbb{T}_2 \otimes E_4) \circ (E_4 \otimes (\mathbb{T}_1 \circ (E_4 \otimes \mathbb{U}))).$$

Theorem 4.3

Two-layer CNN is a sofic shift generated by labeled graph $(\mathcal{B}^{(2)} * \mathcal{B}^{(1)}, \mathcal{L})$.

Outline

N-layer CNN for $N \geq 2$

Theorem 4.4

If $N \geq 2$*, then given* $\left(A^{(i)}, B^{(i)}, z^{(i)}\right)$ *for* $i = 1, \cdots, N$*, then*

1. *The transition matrix of* $(i+1)-$*layer associate with the* i*-th layer can be defined*

$$\mathbb{T}_{i+1,i} = \left(\mathbb{T}_{i+1} \otimes E_4\right) \circ \left(E_4 \otimes \mathbb{T}_i\right).$$

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2. *The transition matrix of* $(i+1)-$*layer associate with the first* $i-$*layers can be recurrence defined*

$$\hat{\mathbb{T}}_{i+1} = \left(\mathbb{T}_{i+1} \otimes E_{4^i}\right) \circ \left(E_4 \otimes \hat{\mathbb{T}}_i\right).$$

where $\hat{\mathbb{T}}_1 = \mathbb{T}_1$.

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where $\hat{\mathbb{T}}_1 = \mathbb{T}_1$.

3. *Introduce* $\mathcal{S} = \{s_{ijk}\}_{0 \leq i,j,k \leq 1}$, *thus* $\mathbb{S}_{i+1}$ *can be recurrently defined from* $\mathbb{T}_{i+1}$.

Composition of sofics

- Thus, we need to study the dynamics of convolution of N-coupled sofic shifts, i.e.,

  $$\Phi_N * \Phi_{N-1} * \cdots * \Phi_1$$

  where Φ_i is a sofic shift for all $i = 1, \cdots, N$.

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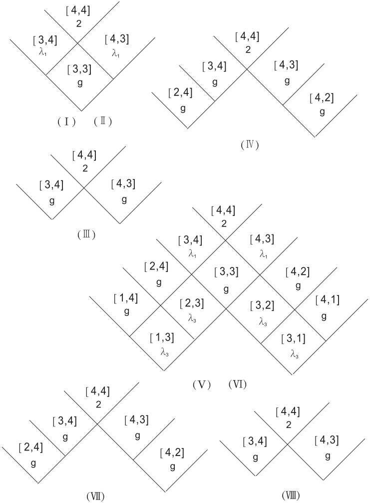
- MCNN is a nature motivation for the convolution of "N-coupled sofic shifts".

Outline

Entropy for one-layer CNN without input

- Symmetric of entropy for 1-D CNN without input.



Entropy for one-layer CNN with input

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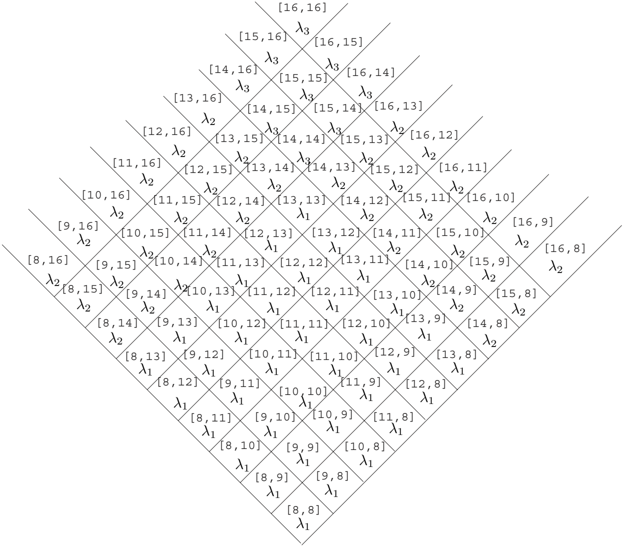
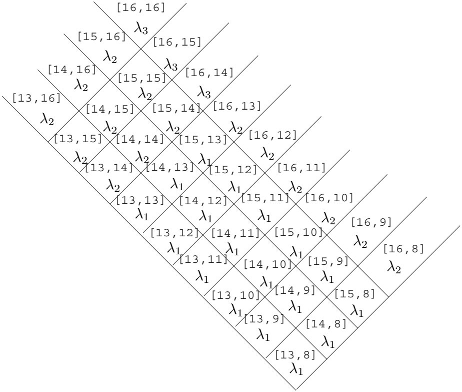


Entropy for 2-layer CNN with input

- Nonsymmetric for N-Layer CNN with input with $N \geq 2$.



	[·,16]	[·,15]	[·,14]	[·,13]	[·,12]	[·,11]	[·,10]	[·,9]	[·,8]
[16,·]	[16,16] λ_3	[16,15] λ_3	[16,14] λ_3	[16,13] λ_2	[16,12] λ_2	[16,11] λ_2	[16,10] λ_2	[16,9] λ_2	[16,8] λ_2
[15,·]	[15,16] λ_2	[15,15] λ_2	[15,14] λ_2	[15,13] λ_1	[15,12] λ_1	[15,11] λ_1	[15,10] λ_1	[15,9] λ_1	[15,8] λ_1
[14,·]	[14,16] λ_2	[14,15] λ_2	[14,14] λ_2	[14,13] λ_1	[14,12] λ_1	[14,11] λ_1	[14,10] λ_1	[14,9] λ_1	[14,8] λ_1
[13,·]	[13,16] λ_2	[13,15] λ_2	[13,14] λ_2	[13,13] λ_1	[13,12] λ_1	[13,11] λ_1	[13,10] λ_1	[13,9] λ_1	[13,8] λ_1

Outline

Recent wroks and Future Plans

1. Two-dimensional MCNN.

Recent wroks and Future Plans

1. Two-dimensional MCNN.
    1. Equivalent to more symbol problem.

Recent wroks and Future Plans

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    2. Use connecting and trace operators in 2-dim patterns generation problem to estimate lower bound and upper bound of output entropy.

Recent wroks and Future Plans

1. Two-dimensional MCNN.
    1. Equivalent to more symbol problem.
    2. Use connecting and trace operators in 2-dim patterns generation problem to estimate lower bound and upper bound of output entropy.
2. Entropy formula and Zeta function formula for MCNN:

$$h(Y_U^{(i)}) = h(Y_U^{(i-1)}) * \cdots * h(Y_U^{(1)})? \text{ for } 1 \leq i \leq N$$

$$\zeta_\sigma(Y_U^{(i)}) = \zeta_\sigma(Y_U^{(i-1)}) * \cdots * \zeta_\sigma(Y_U^{(1)})? \text{ for } 1 \leq i \leq N$$

Outline

1. Introduction
2. One-layer CNN
   - One-layer CNN
   - Without input
   - With input
   - Why sofic?
3. Sofic shift
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   - Entropy for shift space
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Thanks for your attention!